NATIONAL MATHEMATICS MAGAZINE

(Formerly Mathematics News Letter)

Vol. XIX BATON ROUGE, LA., November, 1944 No. 2

$3.00 PER YEAR : SINGLE COPIES 40c.

V

L.

W

G

JO

T.

A.

N.

H.

the
va
(3)
gr

en
or
do

Sm

Sh

SUBSCRIPTION
$3.00 PER YEAR
IN ADVANCE
SINGLE COPIES
40c.

ALL BUSINESS
SHOULD BE ADDRESSED
TO THE
EDITOR AND MANAGER

VOL. XIX — BATON ROUGE, LA., November, 1944 — No. 2

Entered as second-class matter at Baton Rouge, Louisiana.
Published monthly excepting June, July, August, and September, by S. T. SANDERS.

THIS JOURNAL IS DEDICATED TO THE FOLLOWING AIMS: (1) Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values. (2) To supply an additional medium for the publication of expository mathematical articles. (3) To promote more scientific methods of teaching mathematics. (4) To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Every paper on technical mathematics offered for publication should be submitted (with enough enclosed postage to cover two two-way transmissions) to the Chairman of the appropriate Committee, or to a Committee member whom the Chairman may designate to examine it, after being requested to do so by the writer.

Papers intended for a particular Department should be sent to the Chairman of that Department.

COMMITTEE ON ALGEBRA AND NUMBER THEORY: L. E. Bush, W. Vann Parker, R. F. Rinehart.

COMMITTEE ON ANALYSIS: W. E. Byrne, Wilson L. Miser, Dorothy McCoy, H. L. Smith, T. Linn Smith.

COMMITTEE ON TEACHING OF MATHEMATICS: Wm. L. Schaaf, Joseph Seidlin, L. J. Adams, C. N Shuster.

COMMITTEE ON APPLIED MATHEMATICS: C. D. Smith, Irby C. Nichols, H. P. Neal.

COMMITTEE ON BIBLIOGRAPHY AND REVIEWS: H. A. Simmons, P. K. Smith.

COMMITTEE ON PROBLEM DEPARTMENT: Emory P. Starke, N. A. Court.

COMMITTEE ON HUMANISM AND HISTORY OF MATHEMATICS: G. Waldo Dunnington, A. W. Richeson.

COMMITTEE ON BRIEF NOTES AND COMMENTS: Marion E. Stark.

COMMITTEE ON GEOMETRY: N. A. Court, John L. Dorroh, V. Thébault.

The Lure of the Infinite

Man's unsatisfiable curiosity must forever drive him to seek ends of trails that have no ends. Beyond the infinite vista of all the natural numbers must be another number vista, ω, and beyond, ω, the higher vista ω^2, etc., etc. The concept of *class* must go along with that of *class* of *classes*, and this, in turn, with a *class* of *classes* of *classes*, etc., *without end, and,* in spite of the by-brew of contradictions. To the variable of *analysis*, allowed to grow large without bound, that is, without a *last* value, there is made to correspond a reciprocal that grows *small* without a last value, but *not* without a bound, since zero *is* that bound. Thus does the mind's instinct to reach out for the unattainable have a quasi-satisfaction in adopting the convention 1/infinitely-large-number, or $1/\infty = 0$. In geometry, this lure of the *ultimate* is reflected in the purely logical creations, the non-Euclidean geometries, that result from a rejection of Euclid's postulate of parallels. In projective geometry is shown the impulse to handle the infinite. The concept of space elements at infinite distances is used to *mirror*, so to speak, finite distance relations.

These are but briefest intimations of the profound effects upon science that have arisen from the mind's irrepressible instinct to find an utterance for the unutterable, to invent a mode of spanning the infinite. Even if Kronecker could have had his way and could have brought about the elimination of "infinities" from analysis, the mathematics of the "finite" that would have resulted could only have been short-lived, since it would have been subject to critical attack by minds not content with a superficial science. Such minds prefer the profounder approaches to mathematics, even if the paths are beset with paradoxes.

S. T. Sanders.

Skew Curves Setting Up a Null System in Space

By Howard Eves
Syracuse University

1. *Introduction.* One of the most striking projective theorems concerning the twisted cubic is that any three osculating planes of the cubic meet in a point coplanar with their points of osculation. Thus the twisted cubic sets up, in this manner, a null system in space. Let us more generally define an *n-curve* as a curve such that: (A) the points of osculation of all osculating planes to the curve through a point are coplanar with the point, (B) the osculating planes of a set of coplanar points of the curve are copunctual at a point on their plane. An n-curve, then, also sets up a null system in space, and the twisted cubic is but a particular n-curve.

We propose in the following briefly to examine n-curves and some of their properties. We shall employ vector analysis, using non-homogeneous cartesian space coordinates. The vector product of two vectors u and v will be designated by $|u,v|$, and the determinant of three vectors u, v, w by $|u,v,w|$.

2. *Differential equation satisfied by an n-curve.* Consider an n-curve represented by the parametric vector equation $x = x(t)$. In the null system set up by this n-curve, the tangent lines to the curve constitute the self-corresponding lines. Hence these tangent lines are the lines of a linear complex. Now the six (non-homogeneous) Plücker line coordinates of the tangent at x may be taken as

$$|x,x'| \quad \text{and} \quad -x', \tag{1}$$

where the primes indicate differentiation with respect to t. Since the lines constitute a linear complex, the six coordinates (1) must satisfy a linear relation. That is, we must have constant vectors r and s such that

$$|r,x,x'| = s \cdot x', \; r \neq o. \tag{2}$$

We now temporarily define any curve satisfying a differential equation of the form of (2) to be an m-curve.

3. *Equivalence of the class of n-curves and the class of m-curves.* We have already established

Theorem 1. *An n-curve is an m-curve.*

We now establish the converse

Theorem 2. *An m-curve is an n-curve.*

Differentiating (2) we get

$$|r,x,x''| = s \cdot x''. \tag{3}$$

Or, rewriting (2) and (3),

$$(|r,x| - s) \cdot x' = 0 \quad \text{and} \quad (|r,x| - s) \cdot x'' = 0,$$

whence

$$\big| \, |r,x| - s, \quad |x',x''| \, \big| = 0. \tag{4}$$

Now consider a point y on the curve. Then the equation of the osculating plane at y is

$$|x - y, y', y''| = 0.$$

If this is to pass through the point α we have

$$|\alpha - y, y', y''| = (\alpha - y) \cdot |y', y''| = 0,$$

or, from (4),

$$(\alpha - y) \cdot (|r,y| - s) = |\alpha, r, y| - (\alpha - y) \cdot s = (|\alpha, r| + s) \cdot y - \alpha \cdot s = 0.$$

Hence points on the curve whose osculating planes pass through α must satisfy the equation

$$(|\alpha, r| + s) \cdot x - \alpha \cdot s = 0, \tag{5}$$

which is a plane through α. This establishes property (A).

Next consider the plane

$$n \cdot x - b = 0. \tag{6}$$

Let us determine a vector α and a scalar ρ such that

$$|\alpha, r| + s = \rho n, \quad \alpha \cdot s = \rho b.$$

Then an alternative form for the plane (6) is

$$(|\alpha, r| + s) \cdot x - \alpha \cdot s = (\alpha - x) \cdot (|r,x| - s) = 0,$$

which passes through the point α. Let this plane cut the curve in a point y. Then we have, because of (4),

$$|\alpha - y, y', y''| = 0.$$

Thus we see that the point α lies in the plane

$$|x - y, y', y''| = 0,$$

which is the osculating plane at y. This establishes property (B).

4. *Some propective properties of n-curve.*

Theorem 3. *The same null system is set up by an infinite number of n-curves,* namely by all the solutions of the differential equation (2).

Theorem 4. *The class of all n-curves is projectively invariant.*

This is obvious geometrically, for osculating planes, copunctual planes, and coplanar points project into osculating planes, copunctual planes, and coplanar points. The theorem may also be established analytically by subjecting the differential equation (2) to a general projective transformation.

Definition. A point α and the plane P containing the points of osculation of the osculating planes through α will be known as corresponding *pole* and *polar* for the n-curve.

Theorem 5. *Collinear poles correspond to coaxal polars, and conversely.*

Consider two points α and β. Then any point γ collinear with α and β may be taken as

$$\gamma = a\alpha + b\beta, \quad \text{where } a+b=1.$$

By (5) the polar of γ is

$$\begin{aligned}\{ |a\alpha+b\beta,r| + (a+b)s\} \cdot x - (a\alpha+b\beta)\cdot s \\ \equiv a[(|\alpha,r|+s)\cdot x - \alpha\cdot s] + b[(|\beta,r|+s)\cdot x - \beta\cdot s] = 0.\end{aligned}$$

Hence, if the poles are collinear, the polars are coaxal. The converse is similarly established.

Definition. The line joining two poles and the line of intersection of the corresponding polars will be known as *polar lines* of the n-curve. The tangent lines of the curve are self-corresponding polar lines.

5. *Some metrical properties of n-curves.*

Definition. The polar line of a line at infinity will be known as a *diameter* of the n-curve.

Theorem 6. *All diameters are parallel, having direction r.*

Let α and β be any two distinct points on a diameter. Then the polars of α and β are

$$(|\alpha,r|+s)\cdot x = \alpha\cdot s \quad \text{and} \quad (|\beta,r|+s)\cdot x = \beta\cdot s.$$

Since these planes must be parallel we have

$$|\alpha,r| + s = k(|\beta,r| + s),$$

where k is a constant of proportionality. Dotting through by r we find that $k=1$. Hence $|\alpha-\beta,\ r| = 0$. This proves the theorem.

Theorem 7. *The diameter through the point* $|s,r|/r\cdot r$ *is perpendicular to the planes through whose poles it passes.*

For the normal to the polar of $|s,r|/r\cdot r$ has direction

$$\begin{aligned} \big|\,|s,r|/r\cdot r,\ r\big| + s &= \big|\,|s,r|\ r\big|/r\cdot r + s \\ &= [\,(s\cdot r)r - (r\cdot r)s\,]/r\cdot r + s \\ &= [\,(s\cdot r)/(r\cdot r)\,]r. \end{aligned}$$

Since the direction of the normal is proportional to direction r, the theorem is established.

Definition. The particular diameter of theorem 7 will be known as the *axis* of the n-curve.

Theorem 8 *Let p be the shortest distance of a tangent of the n-curve from the axis, and let Θ be the inclination of the tangent to the axis. Then, for all tangents to the curve, p tan $\Theta = (r\cdot s)/(r\cdot r)$, a constant. Conversely, if for some fixed straight line, p tan Θ is constant, then the curve is an n-curve having the fixed straight line for axis.*

The axis passes through the point $|s,r|/r\cdot r$ and has direction r; the tangent line passes through the point x and has direction x'. Therefore, by solid analytic geometry, we have

$$p = \frac{|r,\ x',\ |s,r|/r\cdot r - x|}{[\,|r,x'|\cdot|r,x'|\,]^{\frac{1}{2}}}, \qquad \tan\Theta = \frac{[\,|r,x'|\cdot|r,x'|\,]^{\frac{1}{2}}}{r\cdot x'}$$

Therefore

$$\begin{aligned} p\tan\Theta &= \frac{|r,\ x',\ |s,r|/r\cdot r - x|}{r\cdot x'} \\ &= \frac{|r,x,x'| + (1/r\cdot r)\,|r\cdot x'|\cdot|s,r|}{r\cdot x'} \\ &= \frac{s\cdot x' + (1/r\cdot r)[\,(r\cdot s)(r\cdot x') - (r\cdot r)(s\cdot x')\,]}{r\cdot x'} \\ &= \frac{r\cdot s}{r\cdot r}, \text{ a constant.} \end{aligned}$$

To establish the converse let the fixed line pass through a point α and have direction r. Choose a vector v such that $\alpha = |v,r|/r\cdot r$. Then, as above, we find

$$p \tan \Theta = \frac{|r,x,x'| + (1/r\cdot r)(r\cdot v)(r\cdot x') - v\cdot x'}{r\cdot x'}$$

$$= \frac{|r,x,x'| - v\cdot x'}{r\cdot x'} + \frac{r\cdot v}{r\cdot r} .$$

If this is to be a constant we must have

$$|r,x,x'| - v\cdot x' = k(r\cdot x'),$$

where k is some constant. That is, we have

$$|r,x,x'| = (kr+v)\cdot x' = s\cdot x',$$

where $s = kr+v$. Since $|s,r| = |v,r|$, the theroem is established.

Note. By this theorem a circular helix is an n-curve having the axis of its cylinder for axis.

Definition. The constant $p \tan \Theta = (r\cdot s)/(r\cdot r)$ will be known as the *constant* of the n-curve.

Theorem 9. *All n-curves satisfying the same differential equation* (2) *and passing through the same point x have the same torsion at that point.*

If the torsion is designated by τ we have

$$\tau = -\frac{|x',x'',x'''|}{|x',x''|\cdot|x',x''|} = -\frac{|x',x''|\cdot x'''}{|x',x''|\cdot|x',x''|} . \tag{7}$$

But, by (4),

$$|x',x''| = k(|r,x| - s), \tag{8}$$

where k is a factor of proportionality. Therefore

$$\begin{aligned}(|x',x''|\cdot x''')/k &= (|r,x| - s)\cdot x''' \\ &= |r,x,x'''| - s\cdot x''' \\ &= -|r,x',x''| \quad \text{(from the derivative of (3))} \\ &= -r\cdot|x',x''|. \qquad (9)\end{aligned}$$

Substituting (8) and (9) in (7) we find

$$\tau = \frac{r\cdot(|r,x| - s)}{(|r,x| - s)\cdot(|r,x| - s)} ,$$

a result depending only on r, s, and x. This proves the theorem.

Theorem 10. *Take any point O in space. Let P be any point on an n-curve, and let $OT=1$ be parallel to the tangent at P. Then there exist fixed points R, U, V such that volume $TROP$ = volume $TUOV$.*

Choose O as origin of coordinates and take arc-length as parameter for the n-curve. Then we have constant vectors r and s such that

$$|r,x,x'| = x'\cdot s, \quad x'\cdot x' = 1.$$

Choose any two constant vectors u and v such that $s = |u,v|$. Then we have

$$|r,x,x'| = |x',u,v|, \quad x'\cdot x' = 1.$$

whence the theorem.

6. *Integration of the differential equation.* We proceed, now, to integrate the differential equation (2), thus finding the general n-curve. We will obtain

Theorem 11. *The general n-curve is given, in homogeneous cartesian coordinates, by*

$$x = a(gh' - 2hg') + bh' + cgg' + dg', \tag{10}$$

where g and h are arbitrary scalar functions of one parameter, and a,b,c,d are any constant vectors.

Let the differential equation of the n-curve be

$$|r,x,x'| = s\cdot x'.$$

Now let us rotate the coordinate axes so that the point r falls on the positive x-axis, and the point s on the xy-plane. The differential equation then takes the form

$$p(y\,dz - z\,dy) = m\,dx + n\,dy. \tag{11}$$

We may rewrite (11) in the form

$$d[y(pz-n) - mx] = 2pz\,dy. \tag{12}$$

Now set

$$y = g(t), \tag{13}$$

$$y(pz-n) - mx = h(t). \tag{14}$$

Then (12) becomes

$$2pzg' = h'. \tag{15}$$

Solving (13), (14), (15) for x, y, z we have

$$\begin{cases} x=(gh'-2ng'-2hg')/2mg', \\ y=g, \\ z=h'/2pg'. \end{cases}$$

Hence the homogeneous coordinates of the n-curve (12) are

$$\begin{cases} x=p(gh'-2ng'-2hg'), \\ y=2pmgg', \\ z=mh', \\ w=2pmg'. \end{cases} \tag{16}$$

Because of theorem 4 the homogeneous coordinates of the points on the n-curve in general position may be obtained by subjecting (16) to a general projective trnasformation. Doing this we obtain (10).

7. *Some particular n-curves.*

(a) Taking $g=t^m$, $h=t^{m+n}$, (10) yields

$$x \sim At^{m+n}+Bt^n+Ct^m+D. \tag{17}$$

(b) In (17) set $m=1$, $n=2$ and we obtain

$$x \sim At^3+Bt^2+Ct+D,$$

the *twisted cubic.*

(c) In (17) set $m=1$, $n=3$ and we obtain

$$x \sim At^4+Bt^3+Ct+D,$$

a special *twisted quartic.*

(d) If we take $g=\cos t$, $h=-t+\sin t\cos t$, $a=(0,0-\frac{1}{2}b,0)$, $b=(0,-\frac{1}{2}a,0,0)$, $c=(-a,0,0,0)$, $d=(0,0,0,0)$, we obtain the *circular helix*

$$x=a\cos t, \quad y=a\sin t, \quad z=bt.$$

F-Values for Samples of Four and Four Drawn from Populations Which Are the Sum of Two Normal Populations

By G. A. BAKER
University of California, Davis

A common method for testing for significant differences in yields due to varieties or treatments is to compute Snedecor's F which is the ratio of two variances, one due to differences in varieties or treatments and the other due to "experimental error". The distribution of F with m_1 and m_2 degrees of freedom is

$$\text{(I)} \quad \frac{N\Gamma(m_1+m_2/2)}{\Gamma(m_1/2)\Gamma(m_2/2)}(m_1/m_2)^{m_1/2}F^{(m_1/2)-1}\left(1+(m_1/m_2)F\right)^{-(m_1+m_2)/2}$$

This distribution is given in many places, for instance (4).

The distribution (I) is based on the assumption of a normal distribution of all varieties. It has been shown (3) that only rarely are the fundamental distributions of errors for Agricultural field trials normal. It is thus of considerable interest to investigate what happens to the F-test when the fundamental distributions are not assumed to be normal. Many non-normal distributions can be represented fairly well by the sum of two normal populations. By means of results given in a previous paper (1) and a simple transformation of variables it is possible to write out the distribution of the F-values for any combination of sizes of samples drawn from a non-homogeneous population composed of the sums of normal populations. The expressions obtained are very complicated even for samples of 2 and 2. Some notion of the F-distributions can be obtained by exploratory sampling. Data on the means and variances for samples of four from two such non-homogeneous populations [Chart A, page 341, and Chart B, page 348, (2)] have been presented. Population A is symmetrical and distinctly bimodal, while population B is weakly bimodal and strongly skewed. The purpose of this note is to present the cumulative distributions of F-values for samples of 4 and 4 drawn by throwing dice for the populations A and B of (2) and to compare these with the corresponding normal theory cumulative distribution.

For samples of 4 and 4 $m_1 = m_2 = 3$ and (I) becomes

$$\text{(II)} \qquad \frac{2.5465N\,F^{\frac{1}{2}}}{(1+F)^3}.$$

The cumulative distribution of F is

$$\text{(III)} \qquad \int_0^x \frac{2.5465N\,F^{\frac{1}{2}}}{(1+F)^3}\;dF$$

$$= \frac{2.5465N}{4}\left(\text{arc tan}\,\sqrt{x} + \frac{(x-1)\sqrt{x}}{(1+x)^2}\right)$$

Table 1 gives the cumulative F-distributions for samples from populations A and B and the corresponding values for distribution (III). Moderate and large F-values are more frequent for population B than for a normal parent population. Small and large F-values are less frequent and moderate F-values are more frequent for population A than for a normal population. Population A may be considered as an example of the general class of populations that are symmetrical and platykurtic, while population B may be considered as an example of the general class of skewed populations.

TABLE 1

CUMULATIVE DISTRIBUTIONS OF 524 *F*-VALUES FOR SAMPLES OF FOUR AND FOUR FROM EACH POPULATION A AND B AND THE THEORETICAL CUMULATIVE DISTRIBUTION FOR A NORMAL POPULATION.

x	*Population A*	*Normal Population*	*Population B*
0.5	116	152.8	151
1.0	252	263.5	247
2.0	392	370.5	357
4.0	462	450.4	442
9.0	504	497.0	489
16.0	517	511.7	505
25.0	520	517.4	512
64.0	524	522.1	522
144.0	524	523.4	524

REFERENCES

(1) G. A. Baker, "Random sampling from non-homogeneous populations," *Metron*, Vol. 8, No. 3 (1930.)

(2) G. A. Baker, "The relation between the means and variances, means squared and variances in samples from combinations of normal populations", *Annals of Mathematical Statistics*, Vol. 2, No. 3 (1931), pp. 333-354.

(3) G. A. Baker, "Fundamental distribution of errors for agricultural field trials", NATIONAL MATHEMATICS MAGAZINE, Vol. 16, No. 1 (1941), pp. 7-19.

(4) S. S. Wilks, *Mathematical Statistics*, Princeton University Press (1943) p. 284.

Humanism and History of Mathematics

Edited by
G. Waldo Dunnington and A. W. Richeson

A Ninth Lesson in the History of Mathematics

By G. A. Miller
University of Illinois

18. *Infinity.* The term infinity is used in mathematics with two widely different meanings. According to one of these meanings it is merely a negative term and implies *not finite.* Existence or non-existence is not necessarily involved in this sense of infinity. For instance, if we say that the tangent of 90° is infinite in this sense we do not necessarily imply that the tangent of 90° exists. We merely say that an angle of 90° does not have a finite tangent. Another meaning of infinity is that it is larger than any given finite quantity and this implies existence. In this sense the number of the natural numbers is infinite. Both of these notions of infinity appear in many parts of mathematics and have not always been clearly separated.

Special symbols for infinity appear in the writings of various ancient peoples and some of these symbols were later used to represent finite numbers. For instance, it seems that the ancient Egyptians' symbol for 10,000 was used earlier to represent an infinitely large number. Cf. O. Neugebauer *Geschichte der Antiken Mathematischen Wissenschaften,* page 97 (1934). With the advance of civilization the numbers which were regarded as infinite usually increased in size. This relates however, to the early vague notion of infinity and not to our modern concept of infinity, which seems to have been developed very slowly and was often expressed in doubtful language. Hence its history is difficult as most other history is.

Steps towards the development of our modern concept of infinity appear in the reported views of the Greek Electic School of Southern Italy. In particular Zeno (about 450 B. C.) is said to have asserted that Achilles could not overtake a turtle going at a much slower constant rate of speed than Achilles if the turtle was at a certain distance

ahead of Achilles when he started, since at the time when Achilles reached the place where the turtle was when he started the turtle would have moved some distance ahead, and this process could be repeated an endless number of times. We have here an illustration of the fact that the sum of an infinite number of positive terms may have a finite limit for it is obvious that Achilles would overtake the turtle and there is no difficulty in finding the necessary time when the conditions are definitely stated.

A second well known example of the use of an infinite series of positive terms by the ancient Greeks appears in the works of Archimedes, who flourished about two hundred years after Zeno and used the infinite geometric series

$$1+\tfrac{1}{4}+(\tfrac{1}{4})^2+(\tfrac{1}{4})^3+\cdots$$

to find the area of a segment of a parabola. He noted that the sum of this infinite series is 4/3. These and other examples make it clear very early that a quantity may increase forever and never become very large. It is necessary to note the manner in which a quantity increases in order to reach a definite idea of the limit of its size in the case when it is assumed to increase an infinite number of times. It might have at first appeared probable that if a quantity increase an infinite number of times it would necessarily become large.

The two infinite series to which we have referred are special cases of two geometric series in which the ratio is less than unity. Both, the ancient Babylonians and the ancient Egyptians considered some geometric as well as some arithmetic series but little is known as regards the extent of their knowledge along these lines. The ancient Greeks were interested in infinitely large as well as in infinitely small quantities but proofs based on the consideration of such quantities were not viewed with favor on the part of some of their leaders, including Plato and Aristotle. It has been said that this opposition may have delayed the development of our modern calculus for two thousand years.

The concept of infinity is widely used in calculus. Archimedes used some of the concepts of calculus, including integration, with success even if the development of this subject is largely due to later mathematicians, including Newton and Liebniz who flourished in the seventeenth and the beginning of the eighteenth centuries. The work of Archimedes along this line is explained in a monograph written by him on his method, which was supposed for a long time to be lost but was found by J. L. Heiberg in 1906 and has received much attention since then. It appears as a supplement to the *Works of Archimedes* edited by T. L. Heath (1897) and hence it is now accessible even to those who read only the English language.

One of the earliest proofs of the non-evident existence of an infinite number of elements in a given set appears in the Ninth Book of Euclid's *Elements* and establishes the fact that there is an infinite number of prime numbers by proving that there is always a prime number which exceeds every given prime number. This proof, as well as many of the others which appear in Euclid's *Elements*, seems to be older than these *Elements* and to have been only transmitted by Euclid. It has not been found in any of the pre-Grecian mathematics notwithstanding the fact that it is very elementary and that the pre-Grecian mathematicians sometimes gave proofs of their statements although the contrary has sometimes been asserted.

It might at first appear improbable that an area bounded by circular arcs could be exactly equal to an area bounded by straight lines only. A clear proof of the fact that such an equality is possible was given by Hippocrates (about 440 B. C.) by inscribing a right triangle having two equal sides in a semi-circle whose diameter is equal to the hypotenuse of the right triangle. By means of the so-called Pythagorean theorem it can then be proved that we can construct a surface bounded by a semi-circle and a quarter of another circle whose area is exactly equal to the area of half of this triangle. That is, Hippocrates proved that areas bounded by curved lines may be exactly equal to areas bounded by straight lines. Hippocrates is said to have also written an elementary treatise on geometry.

The use of infinity in mathematics may be explicit or implicit. For instance, the totality of the circles of an infinite number of different sizes, and the infinite number of different shapes and sizes of plane triangles were frequently considered together in each case. The ancient Egyptians and the ancient Babylonians noted already approximate rules for the area of all circles as well as exact rules for the areas of plane triangles, as was noted in the preceding *Lesson*. With the beginning of algebra the same symbol was frequently used for an infinite number of different unknown quantities. The dealing with implicit infinities has naturally increased as mathematics has advanced and greater and greater generalizations have become common in our work. The fact that the ratio between the circumference of a circle and its diameter is the same for all the infinitely different circles naturally interested many thoughtful people long before the nature of this ratio was understood, and it was assumed to be true long before it was proved to be true.

In Euclid's *Elements* there appears the definition that "parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction". On the contrary, the French mathematician,

Desargues (1593-1662) asserted that parallel lines have a point at infinity in common. This introduction of the point at infinity on a straight line makes it possible to say that every two straight lines in a plane have one and only one point in common. When this point is in the finite part of the plane the two lines are not parallel but when it is at infinity they are parallel. By adopting this point of view we can say in elementary algebra that two linear equations in two unknowns have always one and only one solution since each of these equations can be represented by a straight line in the plane. The view of Desargues is however not adopted universally even in modern elementary mathematics.

Just as we may say that two parallel lines meet in a point at infinity we may say that two parallel planes meet in a line at infinity. The use of the points, lines, and planes at infinity has been widely adopted in geometry since the time of Desargues and has often greatly simplified the form of stating geometric theorems. This is especially true as regards the subject of projective geometry as developed during the second half of the nineteenth century in different countries by a large number of different writers. J. V. Poncelet is widely regarded as the creator of projective geometry by means of his *Traité des propriétés projectives des figures*, in which he states that this work was the result of researches which he undertook in the spring of 1813 in the prisons of Russia. It is obvious that under these conditions the work had to be done without good opportunities to consult the earlier literature relating to similar lines of thought.

In about the middle of the seventeenth century a special symbol for infinity, ∞, was introduced by an English mathematician by the name of John Wallis (1616-1703), which has been very widely adopted. It had been used earlier for 1000 and hence its later use for infinity was unusual. This was quite different in meaning, from the very ancient symbol representing infinity in the sense of all, or very large. These symbols were frequently used then to represent any number which exceeds a comparatively small number. Many early mathematicians, such as John Kepler (1571-1630), and Bonaventura Cavalieri (1598-1647), usually restricted their consideration of the infinite to geometric quantities, while John Wallis used this concept also for numerical values. He also employed such formulas as

$$\frac{1}{0} = \infty \quad \text{and} \quad \frac{1}{\infty} = 0,$$

which were widely adopted by other writers. A. L. Cauchy (1789-1857) considered the evaluation of expressions which assume the indeter-

minate form $\infty \cdot 0$, $0^{\circ}, \infty^{\circ}$, 1^{∞} $1^{-\infty}$, but special cases of these forms had been considered earlier by L. Euler (1707-1873) and others.

One of the most fruitful periods in the consideration of an infinite number of elements is that of the development of our differential and and integral calculus. It was noted above that steps towards this development appear clearly in the works of Archimedes but his work along this line was later practically forgotten for a time and somewhat similar considerations seem to have been started anew by Kepler and Cavalieri. The former of these two influential mathematicians of the seventeenth century assumed that the circle may be divided into an infinite number of isoceles triangles, the sphere into an infinite number of pyramids, the round bodies into an infinite number of thin lamini, etc. Cavalieri divided plane figures between two parallel lines into parallel chords and solid figures between two parallel planes into parallel plane sections and then assumed that these areas and these solids had the same ratio to each other as these chords and sections respectively.

Cavalieri wrote an influential work on the geometry of the indivisibles in which a point is assumed to be the indivisible of a line, a line as the indivisible of a surface, etc. Each indivisible was assumed to be able to generate the continium of the next higher order. These ideas were not new at the time of Cavalieri but they were more largely employed by him than by earlier writers. They appear already in the works of Aristotle (384-322 B. C.) whose philosophical views were largely adopted by the medieval writers. As an instance of the reasonings of Cavalieri we may note that he found the sum of the squares of all the lines making a triangle as equal to one-third of the sum of the squares of a parallelogram of the same base and altitude because $1^2+2^2+\cdots+n^2=n(n+1)(2n+1)/6$, and the ratio between this and n^3 is one-third since n is infinite. Hence he concluded that the volume of the pyramid or cone is one-third of the volume of the prism or cylinder of equal base and altitude.

The great usefulness of infinite series was an important factor in the development of the calculus. In many cases infinite series converge so rapidly that their approximate values can be found with a sufficiently close approach to their actual values by considering only the first few terms of the series. Some of these series have a long history. For instance the binomial series was early used as a finite series when the exponent was a small positive integer but was later gradually extended in many ways. In some of these extensions Isaac Newton took a prominent part and his early reputation was *then*, and is sometimes now, unduly augmented by this fact. It is not quite true to say that he discovered "that the elementary binomial theorem can be

extended from positive integral exponents n to arbitrary positive or negative, rational or irrational exponents n". This statement appears on page 475 of the useful volume entitled *What is Mathematics?* by Richard Courant and Herbert Robbins (1941). Many different mathematicians contributed to these as well as to further extensions of the binomial theorem. Cf. Miller's *Historical introduction to Mathematical literature*, 209-213 (1916).

A large number of references relating to the development of the binomial theorem and its extension appears in Tropfke's *Geschichte der Elementar Mathematik*, volume 6, pages 34-5 (1924). On page 41 it is asserted that it results from what precedes that Newton did not possess a satisfactory proof of his theorem but was convinced of the truth of the theorem by the fact that he had always arrived at correct results by its use, and a proof by induction. The quotation of the preceeding paragraph is an illustration of the difficulty which is frequently encountered in brief references to important mathematical advances. These references are, however, usually better than nothing since they may serve as a start towards a broader knowledge of the exact situations and as an incentive to pursue the subject further. For instance the statement that there is only one point at infinity on a straight line raises the question whether $+\infty$ and $-\infty$ should be regarded as the same number, since when the points on a line are assumed to represent real numbers we approach $+\infty$ by going in one direction from 0 and $-\infty$ by going in the other direction. L. Duler directed attention (1755) to the fact that we can go from positive numbers to negative numbers both by going through 0 and by going through ∞.

A distinction is often made between the actually infinite and the potentially infinite, and the existence of the former has often been denied even by those who admitted the existence of the latter. This was done already by Aristotle, while the early writers on religion commonly adopted the view that there is an actual infinity. In a letter written in 1831 to Schumacher C. F. Gauss expressed himself as follows: "I protest against the use of an infinite magnitude as something completed, which in mathematics is never permissible. Infinity is merely a *façon de parler*, the real meaning being a limit which certain ratios approach indefinitely near, while others are permitted to increase without restriction". This letter was reprinted in Volume 8 of the *Collected Works* of Gauss and the views as regards infinity expressed therein have been widely quoted in view of the high estimation of the mathematical contributions of C. F. Gauss.

Galileo (1564-1642) called attention to the fact that it is possible to establish a one-to-one correspondence between the positive integers which are squares of other positive integers and the total number of

positive integers but that there are many positive integers which are not the squares of other positive integers. G. Cantor (1845-1918) later noted many instances of one-to-one correspondences between other sets which are such that a one-to-one correspondence can be established between the elements of the two sets. Through the labors of G. Cantor and his followers there has arisen very extensive literature which is devoted to the study of infinite aggregates. The applications of this subject have been extensively developed and their importance has been widely but at first slowly recognized.

An important difference between the totality of integers and the totality of rational numbers is that the integers can be ordered in a linear set so that each one is smaller than the following one, while between each pair of rational numbers there is an infinite number of other rational numbers. Notwithstanding this important difference it is possible to establish a one-to-one correspondence between the integers and the rational numbers. While it it is difficult to define the terms finite and infinite in a satisfactory manner R. Dedekind (1831-1916), and many later writers, defined an infinite aggregate as one in which the whole may be arranged into a one-to-one correspondence with a part of itself while this is not possible with respect to a finite aggregate. The definition of infinite thus becomes positive while that of finite becomes negative, but the term infinite itself seems to imply a negative definition.

In his *Arithmetica infinitorum* (1655) John Wallis noted that the negative numbers may be regarded as being larger than infinity in in view of the inequalities

$$\frac{1}{4} < \frac{1}{3} < \frac{1}{2} < \frac{1}{1} < \frac{1}{0} < \frac{1}{-1} <$$

At another place he had however also noted that the negative numbers could be regarded as the opposite of positive numbers just as a debt and a credit. In going from positive to negative numbers through the number zero the passage leads to a continuous change in the values of the numbers involved but going from positive to negative numbers through infinity the passage leads to a discontinuity of the numbers involved. This difference was naturally somewhat disturbing to the mathematicians before they had considered carefully the behavior of discontinuous functions such as the tangent of a variable angle at 90°.

In certain considerations it has often been convenient to use various powers of ∞. For instance, if the number of the points on a straight line is represented by ∞ then those of the plane can be represented by ∞^2, and the number of the circles in the plane can be represented by ∞^3, since an infinite number of such circles have the same center.

Hence the plane can be regarded as two dimensional with respect to points, three dimensional with respect to circles, etc. Similarly, our ordinary space can be regarded as three dimensional with respect to points and four dimensional with respect to spheres. It is clear that by considering other figures as elements the number of the dimensions of our ordinary space can be increased indefinitely. Sophus Lie made frequent use of different powers of ∞ in his *Theorie der Transformationsgruppen* (1888-1893), as well as elsewhere. It is commonly asserted that every circle of the plane passes through the same two points at infinity. If this is done the number of points in which two distinct circles of the plane may intersect is increased to four points. By eliminating one of the unknowns from the equations representing two circles in the plane we may obtain an equation of the fourth degree in one of the unknowns whose roots represent one of the coordinates of the points of intersection. In his influential *Propriétés projectives,* first edition 1822, J. V. Poncelet remarked that the circles placed arbitrarily in a plane are therefore not independent from each other as one might at first believe but they have two imaginary points in common at infinity (page 49). These points at infinity are sometimes called ideal points of the plane and they have received much attention.

To a large extent the infinite of mathematics has been endowed with properties which simplify the language relating to the finite by making it possible to avoid the consideration of some special cases. Just as it is found convenient to say that every quadratic equation has two roots even when these roots become equal so it is often convenient to say that every pair of straight lines in the plane intersect in one and only one point. This is however not the only type of the uses of the infinite in mathematics but these uses are very extensive and are often intimately related to the consideration of finite properties and are sometimes an outgrowth of such considerations. Developments relating to the finite and the infinite in mathematics have greatly inspired each other even if many individuals have confined their attention to only one of these types of developments. In 1910 Bertrand Russell asserted that "the solution of the difficulties which formerly surrounded the mathematical infinite is probably the greatest achievement of which our age have to boast". This view is naturally not universally accepted.

The harmonic infinite series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$$

evidently has the properties that its separate terms continually de-

crease in value and have 0 for a limit. The Italian mathematician P. Mongoli (1626-1686) was led by a study of the works of Archimedes to inquire whether an infinite decreasing series has necessarily a finite limit and observed that this is not the case because the given very simple series is actually divergent. This fact is of some historical interest because it indicates that the careful study of infinite series is a comparatively recent mathematical subject, and it should be noted that the harmonic series is also of interest in the theory of music. It may be noted that the harmonic series differs only from that of the natural logarithm of 2 by the sign of the even numbered terms and that the closely related series

$$1+\frac{1}{2^a}+\frac{1}{3^a}+\frac{1}{4^a}+\frac{1}{5^a}+\cdots$$

converges for every value of a greater than 1.

The Teachers' Department

Edited by
WM. L. SCHAAF, JOSEPH SEIDLIN, L. J. ADAMS, C. N. SHUSTER

A Method of Measuring Effectiveness in Teaching College Mathematics

By J. O. HASSLER
University of Oklahoma

The conclusions in this paper grow out of some investigations made in the department of mathematics at the University of Oklahoma relative to efficiency in teaching Calculus. The records of students in the second course in Calculus have not always correlated well with their records in the first course. Since these students were prepared in the first course by different teachers, it seemed worth while to make a study of the success in Calculus II of the various students from Calculus I, grouping them according to the teachers they had in the first course. In this way we get a measure of the relative efficiency of teachers. Since about half of the students remained with the same teacher in Calculus II, the other half going to different teachers, it occurred to me to make a comparison of the grading by these teachers of their own students with the grading of their former students by other teachers in the second course. This study was made over a period of six years.

Table I shows the grade distribution in Calculus II of 677 students who had had Calculus I the preceding semester under eleven different teachers. The four groups of students according to their grades, A, B, C, and D are kept separate, and in four pairs of columns are exhibited their grade distributions in Calculus II. It will be noted that while A students in Calculus I were graded a trifle lower than by the same teachers, the students in each of the other three groups were graded higher by other teachers than by the same teachers they had in Calculus I. This is shown by the line labeled $P_s - P_0$, which measures the excess of the average grade points given students by the same teachers over the grade points by other teachers. On the average the difference in grading, whether by the same teachers or by other

teachers, is trivial. It should be, since students were enrolled at random among all the teachers.

TABLE I

RECORD IN CALCULUS II OF 677 STUDENTS FROM ELEVEN TEACHERS OF CALCULUS I.

Comparison of grades given in Calculus II by same teachers (S) of students in Calculus I with those given by other teachers (O). In computing grade points A, counts 3; B counts 2; C, 1; D, 0; E, $-\frac{1}{2}$; F, -1.

Grades Calculus I:→..	A's: 136		B's: 178		C's: 217		D's: 146	
Grades Calculus II.... ↓	Same	Others	Same	Others	Same	Others	Same	Others
A.......	44	23	13	10	1	1	..	..
B.......	40	7	53	15	14	20	2	5
C.......	9	5	44	11	45	25	15	14
D.......	3	4	13	6	43	27	32	20
E.......	..	1	5	1	8	5	3	6
F.......	..	..	4	3	17	11	27	22
Av. grade points......	2.3	2.2	1.4	1.5	.43	.61	−.12	−.02
$P_s - P_0$..............	0.1		−0.1		−0.18		−0.1	
Average excess of grades by S over that by O:........					−0.07			

Table II exhibits the record of an individual teacher. It will be noticed that in each of the four groups of students (A's, B's, C's, D's) the other teachers graded his students higher than he did, the average excess being 0.41 grade points. This means that on the average his students in Calculus I made nearly a one-half letter higher grade with other teachers in Calculus II than with him. The obvious conclusion is that this teacher is what might be called a "hard" grader. For one

TABLE II

ONE TEACHER'S RECORD

Grades Calculus I:→..	A's: 19		B's: 39		C's: 26		D's: 23	
Grades Calculus II.... ↓	Self	Others	Self	Others	Self	Others	Self	Others
A.......	7	4	4	4	1	..	..	..
B.......	4	2	14	4	4	1	1	2
C.......	2	..	3	2	1	2	3	3
D.......	..	..	2	1	7	4	4	4
E.......	..	..	4	..	2	..	..	..
F.......	..	..	1	..	4	..	5	1
Av. grade points:.....	2.4	2.67	1.43	2.0	0.37	0.57	0	0.60
$P_s - P_0$..............	−0.27		−0.57		−0.20		−0.60	
Average excess:.......	−0.41	(This compares his own grading with others.)						
To compare records of *his* Calculus I students with all students, take his grade point averages minus g.p.a. from Table I.								
The result:...........	+0.1	+0.47	+0.03	+0.5	−0.06	−0.40	+0.12	+0.62

other teacher the record went farther the other way showing that his students were graded by himself more than one half a letter higher than by other teachers. He is doubtless an "easy" grader. Individual records were compiled for all of the teachers. The summarized records of eight teachers are shown in the last column of Table IV. Since no two teachers can teach and grade indentical students in the same course, this is a substitute method of comparing the grading systems of different teachers.

To compare the success of an individual teacher's students with the entire 677 students, we compare the grade point average for each column in Table II with the grade point average for that column in Table I. For example, for A students (in Calculus I) who remained with the same teacher in Calculus II, the average of the grade points is 2.3, which means better than a B average; while for students who were with other teachers the average was 2.2. For the teacher represented in Table II those two averages are 2.4 and 2.67. The excess of the record of his A students above the average record for all students is, respectively, 0.1 and 0.47 grade points. The last line of Table II exhibits this excess for all eight of the columns, two of which (the C students' columns) show negative excess; which means that his C students from Calculus I did not do as well in Calculus II as the average for all C students.

TABLE III

RECORDS OF EIGHT TEACHERS SHOWING FOR EACH THE GRADE POINT EXCESS, AMONG HIS STUDENTS, OVER AVERAGE FOR ALL.

Teacher	*A's*		*B's*		*C's*		*D's*		*Average*
	S	*O*	*S*	*O*	*S*	*O*	*S*	*O*	
1	0.15	0.8	0.35	−0.5	−0.23	0.14	−0.78	−0.38	−0.06
2	0.1	0.18	0.48	−0.17	0.24	0.46	0.27	0.96	0.32
3	0.2	0.47	0.1	0.5	−0.15	0.52	0.52	−0.09	0.26
4	−0.3	−1.07	−0.52	0.0	−0.04	−0.94	−0.35	−0.77	−0.50
5	−0.3	−0.03	0.1	0.5	−0.14	−0.05	0.12	−0.23	0.00
6	0.1	0.47	0.03	0.5	−0.06	−0.04	0.12	0.62	0.22
7	0.33	......	−0.08	−0.5	0.43	−0.48	−0.21	−0.48	−0.12
8	0.7	0.8	0.6	0.5	0.57	0.7	0.12	0.58	0.57

Table III shows the record of this teacher (6) and seven others. The last column, showing the average excess, is the measure of efficiency of the various teachers. The range is from -0.50 to $+0.57$ grade points. This means a range of more than one letter in grading

between the achievements of the students of the teacher numbered 4 and the teacher numbered 8 in the succeeding course.

Table IV is self-explanatory except for the second column. Since some teachers urge students to withdraw early to avoid failure, it was thought best to exhibit the percentages of withdrawals along with the percentages of failures, for an excessive number of withdrawals shou d reduce the number of failures. The third column measures the ability of each teacher's students to succeed in the second course

TABLE IV

PERCENTAGES OF FAILURES OF STUDENTS OF EIGHT TEACHERS IN BOTH CALCULUS I AND CALCULUS II.

Records in Calculus I computed on basis of 1111 students, 116 of which withdrew (W) and only 677 of which enrolled in Calculus II.

Teacher	*Per cent F in I*	*Per cent W in I*	*Per cent F in II*	*Per cent D's I Enroll in II*	*Per cent D's I F in II*	*Average* $P_s - P_0$
All	17.2	[10.4]	14.5	(73.7)	36.6	−0.07
1	14.1	[21.4]	16.9	(62.5)	75.0	−0.19
2	16.2	[11.0]	9.7	(85.3)	22.4	−0.16
3	20.9	[10.8]	10.5	(71.4)	25.0	−0.22
4	19.7	[5.4]	22.5	(68.6)	60.4	+0.57
5	7.9	[15.5]	13.6	(60.9)	25.0	−0.17
6	29.3	[10.8]	13.1	(93.0)	26.1	−0.41
7	2.6	[6.7]	17.7	(67.0)	68.8	+0.31
8	16.4	[10.3]	3.3	(64.3)	16.7	−0.22

in Calculus. It is remarkable that for the teacher numbered 8, who "failed" and "withdrew" fewer than the average number in Calculus I, had only 3.3 per cent of his students failing in Calculus II. The two columns exhibiting the records of the D's are very significant in discovering teachers who are prone to be gratuitous in giving D's to students who are unprepared to go on. It was thought fair to record the percentage of D's who went on and enrolled in the second course for that certainly has a bearing on the percentage of failures. Many of the D's were doubtless students with low grades in other courses who dropped out of school, only the better ones being exhibited in the succeeding column in Table IV.

There are many undetermined variables which affect the records set down here. This is not claimed to be an absolute measurement of teaching efficiency. It does furnish a basis for comparison among

teachers. It was hoped that every teacher in the department would be interested in knowing how he rated in comparison with the department as a whole. The hopes were fulfilled when the matter was presented to the department.

One criticism has been offered with regard to the comparison in Table III. A teacher whose patience, personality, and sympathetic attitude has marked him among the students as one with whom a poor student should enroll will naturally have a poorer type of students to send on to the second course. On the other hand, a "hard-boiled" teacher will draw only the hardiest students, those who perhaps have the native ability and learn the subject in spite of a teacher's unwillingness to answer questions. There will be a difference between the possibilities of success of the two types of students even if taught by the same teacher.

In conclusion it must be said that the measure of a teacher includes many things besides those indicated here. For example, a teacher's sympathetic attitude toward his students, his character, his personality, his value to the institution in extra-classroom activity go a long way toward giving the complete evaluation. All that this study does is to measure in cold figures his effectiveness in producing students who can stand up in a succeeding course. Anyone may form his own opinion as to what part of a teacher's full worth that should be. Finally, there is no reason why this method may not be used as well with high school teachers or teachers of other subjects than mathematics in high school or college if there are two courses in sequence the first of which is a necessary prerequisite for the second.

Unifying Elementary Mathematics by Means of Fundamental Concepts

By Mannis Charosh
New Utrecht High School, N. Y. C.

1. *Introduction.* One of the most serious criticisms which can be made of many modern high school mathematics curricula is that they consist of apparently disconnected units, which, to the student, seem to have little or no relation to one another. To the student, factoring has no relation to squaring a binomial; reducing a fraction has nothing in common with the topic of locus in geometry, with the meaning of identity in algebra, or with similarity of triangles. These varied topics can be related to one another with the help of certain fundamental unifying concepts. Some may prefer to use the term "generalize", others may say "enrich"; the author prefers the term "unify".

The desire to unify mathematics springs from a powerful motivating force which has made itself felt throughout the history of learning. Euclid's object was to unify the scattered results in geometry which he had gathered together. Linnæus' work in biology, Mendeleef's work in chemistry, the work of Galileo, Newton, and Einstein in physics, and the work of many pioneers in philosophy, psychology, the social sciences and countless other fields have all resulted from the desire to unify the materials of their respective fields.

Such an approach to the materials of secondary mathematics can serve to improve the student's understanding of the subject. He will see relationships hitherto unsuspected. He will avoid errors that he might otherwise fall into because of confusion of concepts, as, for example, confounding quantity with operation. He will enjoy the subject more because it will satisfy an inner desire (subconscious, perhaps) to see its various parts in relation to the whole In the present paper, the following concepts are discussed: (1) the concept of operation; (2) the fundamental assumptions of algebra; and (3) transformations and invariants. It is the object of this paper to show, by specific reference to high school classroom procedures, where these concepts can be used.

2. *Operations.* In the conventional elementary algebra course the opportunity arises to discuss almost all the operations which occur in the four years of high school mathematics. Most syllabi refer

specifically to the four fundamental operations, viz., addition, subtraction, multiplication and division. From a consideration of these as a class, the idea of an operation can be conveyed as the carrying out of some rule. In these cases the operations are each indicated by a symbol called an *operator*. An operator is therefore a symbol which tells us what rule to apply to the numbers to which the operator is applied. With this idea established, other methods can be shown for indicating operations. Thus some operations are indicated by the placement of numbers with respect to each other. For example: (a) in the denary scale, 53 means: "add 5 *tens* to 3 *ones*"; (b) 7^2 means multiply 7 by 7. We call this latter operation "raising to a power". A third type of operation also finds its place in elementary algebra, viz., that which applys to a single quantity instead of to two or more. For example: (a) Radicals: $\sqrt{5}$ means "find a number whose square is 5"; (b) Trigonometry: cos 30° means "find the ratio of two certain sides in a right triangle of which one angle is 30°". All of these operations occur in the first year of most algebra courses. Other operations which occur in later courses are: (a) Logarithms: log 53 means "find the power of ten which is 53"; (b) Several from advanced algebra, such as ${}^nPr$, ${}^nCr$, d/dx, $n!$, etc.

One advantage of the concept of operation is that it provides motivation for teaching the mechanical manipulation of the various kinds of quantities which students encounter. For example, in elementary algebra, after the concept of the four fundamental operations on integers is discussed, and after literal quantities are introduced, the question may be raised: how do we define these operations for literal quantities? This leads to the introduction of the concept of similar terms for addition and subtraction, the concept of exponents for multiplication, and of division of literals for division. The student will encounter a great many kinds of quantities in his mathematical career: positive integers, negative integers, fractions, literal quantities, irrationals, powers, imaginaries, complex quantities, and even vectors. If each kind of quantity is treated in this way, the student eventually raises the question himself as to how operations are defined for them. For example, it will be found that a bright elementary algebra student will ask how the operation of raising to a power is defined for negative or fractional powers. In intermediate algebra the question will be raised: how is the operation of finding a logarithm defined for negative quantities? Of course, you may have to put off some of these questions by saying that these things are studied either in later high school mathematics courses or in higher mathematics. But you have at the same time given the students a glimpse of new number worlds which arouse their curiosity and increase their interest in the subject.

When defining multiplication for signed numbers it has been the practice of some teachers to justify the laws by using certain concrete examples. The writer has heard two or three talks on such methods, and in none of them was any mention made of the fact that the laws of signs are definitions. If we teach these laws only through concrete illustration, we are training students in just the kind of loose thinking that we wish them to avoid; for the students get the impression that the rules of signs are derived from the illustration. This would not be so bad if it were not that we then proceed to interpret signs in other concrete situations, e. g., north and south, gain and loss, etc. Of course, just the reverse is true. The laws of signs are defined* and can be interpreted for those physical phenomena whose laws are isomorphic with the laws of signed numbers. It is by no means intended to imply that these concrete illustrations should no longer be used. On the contrary, continue to use them—motivate the formal definitions of multiplication of signed numbers with them. But it must be made clear to the students that they are only concrete illustrations and not proofs of the laws.

In plane geometry an introductory unit can be based on applying the operations of addition and subtraction to lines and to angles, and the operations of multiplication and division to lines by integers. Later the operations of multiplication and division applied to lines alone can be used to motivate constructions like ab/c, or square root for $\sqrt{ab}$. Still later, the operation of multiplication alone of lines can serve as an introduction of areas.

In intermediate algebra the introduction of imaginary quantities arises very naturally, and students are not shocked by the idea of the square root of a negative number. The same is true for the introduction of negative angles in trigonometry. Other examples will occur to the reader. An important advantage of teaching the concept of operation is that students will profit by the distinction made between quantities and operators. We all known the common mistake of this type made in trigonometry: $\cos(30°+60°)=\cos 30°+\cos 60°$. The student is tempted to make this mistake because he thinks of the symbol *cos* as a quantity instead of as an operator; and he then proceeds to apply the distributive laws which would in the former case be valid. This leads us naturally to the laws which apply to operators, viz., the fundamental assumptions of algebra.

3. *The Fundamental Assumptions of Algebra.* In a first-term

*In the modern development of algebra, it is proven from a particular set of assumptions that $(-1)(-1)=+1$. (See, for example, *A Survey of Modern Algebra*, by Birkhoff and MacLane, p. 5). But it is wiser in elementary mathematics to consider it a definition.

course of integrated mathematics, the use of the parenthesis may be introduced by some such problem as this: The dimensions of a rectangle are 2 and 5; if the length is increased by some quantity (say x), represent the new area. The students realize from previous discussions that operations must be defined every time a new kind of number is introduced, or whenever a new combination of familiar numbers is introduced. They know how to apply the operation of multiplication to integers by integers, literals by literals, and integers by literals. They also know how to represent the sum of an integer and a literal, e. g., $5+x$. But here is a new situation: they are required to multiply an integer by the sum of an integer and a literal. This requires a new definition of multiplication. We therefore teach the representation as $2(5+x)$ and define the product to be $2\cdot 5+2\cdot x$. The two operations involved are multiplication and addition. In words, we state the assumption (or law, as it is commonly called) thus: *multiplication is distributive with respect to addition.* From this example the generalized definition may be stated: $a\cdot(b+c)=ab+ac$, provided that a, b, c are quantities, not operators. As a more familiar example that this distributive law works, an instance of ordinary multiplication of integers can be given: thus

$$4\cdot 53=4(50+3)=4\cdot 50+4\cdot 3=200+12=212.$$

Indeed, the theory upon which our familiar algorithm of integral multiplication is based can be explained at this point. The full explanation of the theory will require the use of the commutative law of multiplication to which reference will be made later. After giving this illustration, it should be pointed out that the students have been using the distributive law of multiplication with respect to addition ever since they learned to multiply, only the law had never been put into words before. They had simply been taught a mechanical procedure which automatically took care of the carrying out of the law.

The following is a typical instance of what has happened in an intermediate algebra class where the author has discussed this law. The example on the board involved the expression $3(x+2)$.

First student writes: "$3x+2$".
Teacher: "What is wrong?"
Second student: "It should be $3x+6$".
Teacher: "What mistake did he make?"
Second student: "He didn't distribute the multiplication".
Teacher: "Why should he distribute the multiplication?"

Second student: "Because multiplication is distributive with respect to addition".*

With this as a starting point it becomes natural, because of the student's familiarity with the concept of operator, to try other combinations. Is multiplication distributive with respect to subtraction or multiplication, or division? In each case it must be made clear that the answer is a definition,† and that numerical examples are illustrations which show that they have been using these laws (or definitions) regularly without having had their special attention drawn to the fact that the laws were involved. Thus an example like this:

$$\left.\begin{aligned} 3(8-2) &= 3\cdot 8 - 3\cdot 2 \\ &= 24-6=18 \end{aligned}\right\} \text{ compared with } 3(8-2)=3\cdot 6=18$$

illustrates, but does not prove, that multiplication is distributive with respect to subtraction. Factoring should be taught as the reverse of the distributive laws of multiplication with respect to addition and subtraction. Thus, when the common error is made in the example: $P+Prt=P(rt)$, application of the distributive law shows the error at once.

For convenience in the remaining discussion, reference will be made to this propositional function: *Operation A is distributive with respect to operation B.* Listed below are the operations which give significance to this function (for our purposes):

Operation A	*Operation B*
1. multiplication	1. addition
2. division	2. subtraction
3. raising to a power	3. multiplication
4. finding a root	4. division
5. finding a logarithm	
6. finding a trigonometric function	

We shall take all combinations of A and B, state the truth or falsity of the resulting proposition, and give illustrations of their applications to classroom situations.

A1,B1 and **A2,B2.** These have already been discussed above.

*H. Sitomer has brought my attention to a wording which his students find easier to recall and use, viz.: "When multiplying a sum, distribute the multiplier". The other distributive laws can be similarly reworded.

†The question of using the term definition, theorem, law or assumption is not important at this elementary level of algebra. It is logically satisfactory to think of them all as assumptions, or even as definitions. For example, the distribution of finding a square root with respect to multiplication may be thought of as defining multiplication of radicals.

A1,B3. This proposition is false. To clear of fractions, a common mechanical procedure is as follows:

$$\overset{\overset{1}{\smile}}{\tfrac{2}{3}}(x\overset{\overset{3}{\smile}}{+}3)=\overset{\overset{3}{\smile}}{5}<3.$$

Students frequently make the error, given in the illustration, of putting multipliers over both terms of the product. In explaining the error, the statement should be made that multiplication is not distributive with respect to multiplication.

A1,B4. This proposition is false. When students say that $3(2/5)=3\cdot 2/3\cdot 5$, they are assuming the truth of the proposition.

A2,B1. This proposition is true. When a student makes the following error:

$$\frac{\cancel{3}x+5}{\cancel{3}}=x+5,$$

he should be corrected with the statement that he did not distribute the division with respect to the addition. Students will discover for themselves that this fraction can be transformed into $x+(5/3)$ when the division *is* distributed. When the roots of a quadratic equation turn out:

$$\frac{4\pm 2\sqrt{2}}{2},$$

students will not be tempted to divide the denominator into the 4 only, but will distribute the division and obtain $2\pm\sqrt{2}$. In trigonometry, the expression $(1+\sin x)/\cos x$ can be transformed into $\sec x+\tan x$ immediately by distributing the division.

A2,B2. Similar to above.

A2,B3. This proposition is false. In the example $6(2x+4)/2$, students are frequently tempted to divide the 2 into both terms of the product in the numerator. The error is explained by stating that division should not be distributed with respect to multiplication.

A2,B4. This proposition is false. No frequently occuring application comes up in the classroom.

A3,B1(2). These propositions are false. When students make the common error $(x+y)^2=x^2+y^2$, they are assuming the truth of this proposition. From the point of view of factoring, the same error is made: $x^2+y^2=(x+y)(x+y)$. When solving equations like $2x-5=0$, a frequently occuring error is to "square both sides" and obtain

$9x+25=0$. Here "raising to a power" was distributed with respect to subtraction.

A3,B3(**4**). These propositions are true. For example $(xy)=x^2y^2$, and $(x/y)^2=x^2/y^2$. It is important to point out that the distribution is indicated by the presence of the parenthesis. Otherwise the students will distribute incorrectly in an example like this: evaluate $2x^2$ when $x=3$; answer: $6^2=36$. Another application occurs in the problem $3\sqrt{x}=2$. Square both sides, distributing with respect to multiplication, and we obtain $9x=4$. Still another useful example occurs in geometry problems of this type. Find the ratio of the areas of two similar triangles in which two corresponding sides are 36 and 24, respectively; thus, $A_1/A_2=(36)^2/(24)^2$. Application of the distributive law gives us: $A_1/A_2=(36/24)^2=(3/2)^2=9/4$. Otherwise students will square 36 and 24, resulting in more work and more chance for error.

A4,B1(**2**). These propositions are false. They are frequently assumed to be true by students. In the Pythagorean relation: $c^2=a^2+b^2$, the great temptation is to take the square root of each term. Application of the proper distributive law shows this to be impossible. Another common problem is the following: find the dimensions of a rectangle whose perimeter is 34 and whose diagonal is 13. The equations are $2x+2y=34$, and $x^2+y^2=169$. The temptation is to say: $x+y=13$. In the equations of the conic sections $x^2+y^2=25$, $4x^2+9y^2=36$, and $4x^2-y^2=4$, the same temptation occurs. In trigonometry, students frequently try to change $\sin^2 x+\cos^2 x=1$ to $\sin x+\cos x=1$; or $\sin^2 x=1-\cos^2 x$ to $\sin x=1-\cos x$.

A4,B3(**4**). These propositions are true. They may be thought of as defining the multiplication and division of radicals. Thus $\sqrt{ab}=\sqrt{a}\cdot\sqrt{b}$ states a distributive law and defines multiplication of radicals. From it, we justify the transformation $\sqrt{18}=\sqrt{9\cdot 2}=\sqrt{9}\cdot\sqrt{2}=3\sqrt{2}$; similarly for division. In plane geometry, problems involving mean proportionals may lead to an equation of this kind: $x^2=4\cdot 9$. To solve for x, take the square root of each side, distributing with respect to multiplication on the right side. When large numbers occur, this method is particularly useful.

A5,B1(**2,3,4**). These propositions are false. The errors that students make in this connection are quite obvious and require no further comment.

A6,B1(**2**). These propositions are false. The teacher of trigonometry will recognize the common error: $\sin(x+y)=\sin x+\sin y$, etc. However, care should be taken not to say that "functions are not dis-

tributive with respect to addition", because in a problem like: $\cos x(\tan x+\cot x)$, students will be afraid to multiply $\cos x$ by each term in the parenthesis.

A6,B3(4). These propositions are false. No practical illustrations arise in classroom work.

Before leaving the subject of the distributive laws it would be well to show how easy it is to learn which are true and which are not true. If we refer to B1 and B2 as group I, to A1 and A2 as group II, and to and to A3 and A4 as group III, then we may say that each group is distributive with respect to the preceding group, but to no others. All other combinations are false. Thus we have summarized 36 true and false distributive laws in a single statement.

We come now to the commutative laws. The most frequent reference to this law occurs in regard to the expression $x+y$ and $x-y$. In the first case we may change $x+y$ into $y+x$ because addition is commutative. In the second case we may not change $x-y$ into $y-x$, because subtraction is not commutative. These facts are pertinent in the common problems of transforming $(x+y)/(y+x)$ and $(x-y)/(y-x)$ into reduced forms. Another occasion requiring its use occurs when adding similar terms. Terms like xy and yx are similar because multiplication is commutative. In trigonometry, occasion will be found to refer to the commutative law when proving identities. Frequently the analysis of an identity will come down to a step like this: $\sin x \cos x = \cos x \sin x$. This is an identity because multiplication is commutative.

Finally we come to the associative law, which is important in discussing the definitions of operations on more than two quantities. If we are to stick to the program of never operating in a new situation unless the operation is defined for that situation, then it is important in expressions like $x+y+z$ and xyz to refer to the fact that addition and multiplication are associative. In fact, certain short cuts in arithmetic are based on these laws and the commutative laws. The product $9 \cdot 14 \cdot 5$ can be calculated quickly if thought of as $9 \cdot (14 \cdot 5)$, whereas the average student would do it the longer way: $(9 \cdot 14) \cdot 5$. In adding a column of figures, the accountant's device of grouping digits whose sum is ten is based on the associative and commutative laws of addition. As a final illustration: in elementary algebra, when teaching that the product $3a \cdot 2a = 6a^2$, reference should be made to the application of the commutative and associative laws.

4. *Transformations and Invariants.* In higher mathematics these terms have precise meanings which require rigorous treatment. Such a treatment would be impossible in secondary mathematics. In the

high school these terms can be treated from the point of view of appreciation. Yet their application can be made sufficiently precise to serve the purposes stated in the introduction to this paper. By a transformation we will mean "a change according to some rule". For example, in arithmetic we transform $4/8$ into $1/2$ according to certain rules of division. In geometry we transform a triangle by moving it to a new position according to a rule expressed in one of the postulates. After each transformation, certain properties have not changed. These we will call *invariant* properties. For example, after transforming $4/8$ into $1/2$, the numerical value remains invariant. After transforming a triangle by moving it to a new position, the measures of corresponding parts remain invariant.

It is possible to apply these two ideas throughout the elementary mathematics curriculum. We begin with arithmetic. A basic problem in arithmetic is that of determining transformations which will be invariant with respect to the numerical value. We will call these simply "invariant transformations". Thus, dividing or multiplying the numerator and denominator of a fraction by the same non-zero quantity is an invariant transformation. It is because these transformations are invariant that we may transform $1/2+1/4$ into $2/4+1/4$; or $3/5$ into $15/25$; or $5/(2+\sqrt{3})$ into $5(2-\sqrt{3})/[2^2-(\sqrt{3})^2]$. Carrying out the distributive laws is an invariant transformation. That is why we may transform $\sqrt{32}$ into $\sqrt{16\cdot 2}$ into $4\sqrt{2}$, or $(36)^2/(24)^2$ into $(36/24)^2$ into $(3/2)^2$. It is the fact that these transformations (as well as others) are invariant transformations that makes them so useful to us.

In algebra we begin with the topic of the formula and the study of variation. Each problem of this kind is nothing more than a statement of what is transformed and what is invariant. For example: "in the formula $R=E/I$, with E fixed, what happens to R as I increases?" The problem states that in the transformation involving the changing of I, E is to be invariant. A basic problem in algebra, analogous to the one mentioned above in regard to arithmetic, is that that of finding transformations on algebraic expressions whose invariant property is that equal substitutions will yield equal results. For example, if $2x+3x$ is transformed into $5x$, any substitution for x will make $2x+3x$ equal to $5x$. This is, of course, the concept of algebraic identity, which it is important to contrast with the concept of equation. The answer to the question: "may $3x+5$ be transformed into $2x+3$ for any value of x?" will make this contrast stand out so much more clearly. As in arithmetic, it is this invariant property of the algebraic identity which makes it so useful. That is why in $a^2-b^2=(a+b)(a-b)$ we may substitute $x+y$ for a, and z for b, and so factor $x^2+2xy+y^2-z^2$ into $(x+y+z)(x+y-z)$. That is why an identity like that expressed

in the binomial theorem means so much although stated so compactly. The reader will think of other illustrations similar to these.

When presenting the solution of equations one usually discusses the meanings of terms like *equation, root, satisfy, solve, check,* etc. The first equations should be solved by trial substitutions. Difficult cases will motivate the need for an algorithm. The algorithm for solving equations can be further motivated by posing the question: "What transformations on an equation will keep the numerical value of the unknown invariant?" After such an approach, division by zero on both sides of an equation, or taking the square root of both sides, can be treated as transformations not having the invariant properties required of algorithms for solving equations. A whole class of instances occurs as a result of the definition of new algebraic quantities. One example will suffice. Thus the transformation $\sqrt{x}=x^{\frac{1}{2}}$ is an invariant transformation because both are defined in exactly the same way; that is, each is a quantity which when squared will give x as a result. Fractional, negative and zero exponents may be treated this way. Other algebraic illustrations are as follows. If the equation $x^2-4x+k=0$ is transformed by changing the value of k, the sum of the roots of the equation is invariant. If $y=mx-1$ is transformed by changing the value of m, the invariant property is that all the resulting straight lines are parallel to a fixed line whose slope is m. Obviously, these examples by no means exhaust the subject of algebra.

Elementary geometry is rich in examples of the applicability of transformations and invariants. In fact, geometric models with movable parts can be made for the express purpose of investigating intuitionally the invariant properties of a geometric figure under transformations which are possible by the motion of these parts. For example, a parallelogram constructed of four thin rods hinged at the vertices and with rubber bands acting as diagonals* is subject to the transformation of its angles and of its diagonals. Some of the many invariants are these: (1) the lengths of the sides; (2) the sum of the four angles; (3) the sum of any pair of adjacent angles; (4) the parallelism of the opposite sides; (5) the fact that the diagonals bisect each other; (6) the equality of opposite angles; (7) the equality of opposite sides. Some of these are statements of formal theorems. The remaining illustrations from geometry are tabulated below. Needless to say, the list does not exhaust the possibilities.

In trigonometry, the most frequently occuring application is in connection with trigonometric identities. Here the treatment can be exactly like that of algebraic identities. The numerical value of

*See M. Charosh, "The use of models in teaching plane geometry," *High Points*, February, 1932.

Transformation	*Invariant property*	*Related topic*
A geometric figure is moved to a new position.	Magnitudes of corresponding parts.	Postulate used in superposition theorems.
A line moves parallel to its initial position.	Constancy of corresponding angles made with a fixed transversal.	Parallel line theorems.
A point moves along a circle.	Distance from moving point to the center of the circle.	Locus; also, proving existence of the center.
A point moves along the perpendicular bisector of a given line segment.	Equality of distances from the point to the ends of the segment.	Locus (all the locus theorems can be treated similarly).
The vertex of triangle moves along a line parallel to its fixed base.	Area of the triangle.	Transformation of polygons.
A point moves along the arc of a circle.	Constancy of the angle subtended at this point by two given fixed points on the circle.	Angle measurement in a circle.
The sides and angles of a polygon change, while the number of sides remains the same.	(1) number of sides; (2) sum of angles of of polygon.	Theorems on angle-sums in a polygon.
The sides of a polygon are stretched proportionally.	Constancy of corresponding angles.	Properties of similar triangles.
The sides of a right triangle are stretched while the angles remain constant.	Constancy of ratios of corresponding sides taken in pairs.	Definitions of trigonometric functions.*
The diameter of a circle is changed.	Constancy of the ratio of the circumference to the the diameter.	Value and use of π.*

*It is because the ratio *opp./adj.* in a right triangle with a given angle is constant, that we can give it a special name; in this case, the *sine*. tan

*The general concept of mathematical constant may be discussed in advanced classes.

$\sin^2 x + \cos^2 x$ is invariant under all transformations on x. To show that the word *all* may be used will require generalizing functions of angles to include any positive or negative angle. Special treatment will be necessary for the critical angles: 0°, 90°, 180°, 270°. As in algebra, some invariant transformations will follow from equivalence of definitions. For example, the invariant equality of $\tan x$ and $1/\cot x$ under all transformation of x follows for this reason.

Before leaving this subject a word must be said on the important question of when to make a transformation. If the answer to a problem turned out to be $a^2 - b^2$, the inevitable question by students would be: "Shouldn't we factor that?" On the other hand, if the procedure in solving the same problem happened to result in the answer $(a+b)(a-b)$, the same pupils would ask: "Shouldn't those factors be multiplied together?" To these students mathematics is a game of transformations. Given a mathematical expression, they will automatically transform it into something else, without being asked to do so. The blame lies with those textbooks and teachers who require that answers be "simplified"—without a specific purpose. It should be made clear that transformations are not ends in themselves, but are performed with a purpose. In the author's classes it has taken the form of a motto: "Transform only with a purpose". The purposes are made clear by the statement of the problem. Thus to add 1/2 and 1/4 it is more convenient to use 2/4 than 1/2. If "simpler" means "more convenient", then in this instance 2/4 is *simpler* than 1/2. If, on the other hand, the problem is to multiply 54 by 1/2, it is more convenient to use 1/2 than 2/4. In this case then, 1/2 is *simpler* than 2/4. If the answers of the entire class are to be checked quickly for verification as to accuracy, then some standard form might be agreed upon. For this reason it is convenient to have each student reduce a fractional answer to its lowest terms. The so-called *standard* or *canonical* forms in mathematics, whether they be in the theory of the quadratic equation or in differential equations, are no more than a convenience, and are not to be considered as ends in themselves.*

The aforementioned motto serves a double purpose: first to syllabus makers, who can use it as a guide when deciding what transformations to include. They may decide, for example, that reduction of fractions beyond a certain degree of complication will never serve an important purpose in the future mathematics of the high school student. A second purpose is as motivation to students who frequently ask (with disgust in their voices) why they must bother to learn so many manipulations. The author has found that students are *always* satis-

*But of course the purpose of a transformation *can be* to drill students in learning to perform the transformations with skill.

fied with the explanation that we learn as many transformations as possible so that in future problems we can perform those transformations which help us do the problem in the most convenient way. Thus it is easier to divide $(x+y)(x-y)$ by $x+y$ than it is to divide x^2-y^2 by $x+y$; therefore we transform x^2-y^2 into $(x+y)(x-y)$.

5. *Conclusion.* In closing, it need only be pointed out that the fundamental concepts here discussed have been applied to practically every mathematical topic that the high school student meets. The language of transformations and invariants can serve to unify such varied topics as reducing a fraction, the meaning of identity in algebra and its applications, the similarity of triangles, the concept of locus, the value of π, and so on. Finally, it may be noted that there are significant implications for extending the applications of these concepts and of others (e. g., that of limit) to mathematics at the college level.

Brief Notes and Comments

Edited by
MARION E. STARK

10. *Notes From a Freshman Class Room.* I much prefer to teach trigonometry to students who have never heard of sines and cosines. A class in which a few students have had a high school course in trigonometry is a handicapped class in college trigonometry.

In proving identities, I write down the equality, precede it by "To Prove," and draw a line under it. Then I write down whichever side I want to manipulate, and proceed until I obtain the expression on the other side of the equation. This avoids the temptation to use both sides, and such awkward conclusions as $1=1$.

I have never seen the abbreviation CIS A for $\cos A+i \sin A$ in print, but some of my students are familiar with it. I think it is useful in emphasizing that the polar form of a complex number is not merely an expression involving trigonometric functions but is the modulus times a specific function of the amplitude. I always give my classes some problems such as $-3\ (\cos 30°-i \sin 30°)$ to transform into polar form. Likewise $5\ (\sin 20°+i \cos 20°)$ is considered at least on the board. I ask if $2\ (\cos 17°+i \sin 29°)$ is in polar form, and also if it is a complex number.

In teaching polar coordinates, I believe in the hard way of trying to make the students think in terms of distance and angle. I do not mention the possibility of transformation into rectangular coordinates until we are finishing the chapter on polar coordinates. It is my desire that an equation in polar form shall mean something to my students when they look at it. Until I taught polar coordinates I always thought of them as an awkward and unnecessary way of handling curves, and far inferior to Cartesian coordinates. I hope my way of teaching does not have a similar effect on my students, although of course it may.

When a student asks "What's the use or application of this?", what he really means is "I don't like this. Why bother with it?"

Is there such a thing as a really satisfactory text book? Even when an author is using his own book do you suppose he is completely satisfied? Could this dissatisfaction with a fixed, static, ar-

ranged text be the so-called divine kind of discontent that seeks growth toward greater perfection? Are our quarrels with arrangement or order of topics due to the fact that what we desire (and supposedly possess ourselves) is a simultaneous knowledge which can only be imparted sequentially? We are forced to choose a linear path connecting points that are imbedded in multi-dimensional space. There is no one best path. Brachistochrones do not always turn out to be the geodesics of the manifold.

I taught the traditional arrangement of freshman mathematics, then for five years taught unified mathematics under a crusading head of department and for the past four semesters the traditional arrangement. My grand conclusion is that both are good ways. The two factions who get worked up about how mathematics ought to be taught remind me of zealous members of two denominations of a church. They both feel they have found the one best way of seeking the ideal and no rational argument is possible. However, there are a lot of people who could go to either church and be perfectly satisfied, and get as much from one as the other.

Illinois Institute of Technology. RUTH MASON BALLARD

Problem Department

Edited by
E. P. STARKE and N. A. COURT

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscripts be typewritten with double spacing. Send all communications to EMORY P. STARKE, Rutgers University, New Brunswick, N. J.

SOLUTIONS

No. 540. Proposed by *N. A. Court*, University of Oklahoma.

The planes (M_a), (M_b), (M_c) passing through the Monge point of a tetrahedron (T) and perpendicular to the bimedians m_a, m_b, m_c cut the bimedians m_b and m_c, m_c and m_a, m_a and m_b in the points P_b and P_c, Q_b and Q_a, R_a and R_b. If G is the centroid of (T) and k^2 the sum of the squares of the edges of (T), we have

$$GQ_a \cdot GR_a + GR_b \cdot GP_b + GQ_b \cdot GP_c = k^2/16.$$

Solution by *L. M. Kelly*, U. S. Coast Guard Academy.

The planes (M_b), (M_c) meet the bimedian m_a in the points Q_a, R_a. By Court's theorem* these two points are harmonic conjugates with respect to the extremities of m_a, hence

*N. A. Court, *Modern Pure Solid Geometry*, p. 51, Art. 168 and p. 70, Art. 234. Macmillan, 1935.

$$GQ_a \cdot GR_a = m_a{}^2/4.$$

Adding this equality to the two analogous ones relative to m_b and m_c we obtain

$$GQ_a \cdot GR_a + GR_b \cdot GP_b + GQ_c \cdot GP_c = (m_a{}^2 + m_b{}^2 + m_c{}^2)/4.$$

Now the right hand side of this equality is equal to $k^2/16$,† hence the proposition.

†*ibid.*, p. 56, Art. 186.

Similar solutions by *Joseph S. Guérin* and *Paul D. Thomas*.

No. 541. Proposed by *N. A. Court*, University of Oklahoma.

In the face ABC of a tetrahedron $D-ABC$, find a point M such that

$$\text{area } MBC : \text{area } MCA : \text{area } MAB = \text{area } DBC : \text{area } DCA : \text{area } DAB.$$

I. Solution by *Hyman J. Zimmerberg*, University of Chicago.

Let p, q and r denote the areas of triangles DBC, DCA and DAB, respectively, and let $s=p+q+r$. If K is the area of triangle ABC, then the areas of triangles MBC, MCA and MAB must necessarily be pK/s, qK/s and rK/s, respectively. Let a, b and c denote the sides of triangles ABC opposite to the vertices A, B and C, respectively. In the plane of ABC construct a line parallel to AB at a distance $2rK/sc$, which meets the triangle. Similarly, construct another line parallel to BC at a distance $2pK/sa$, which meets the triangle. The two lines thus drawn will intersect in M. A similar third line parallel to AC at a distance $2qK/sb$ will necessarily also pass through M.

II. Solution by *J. S. Guérin*, student, Catholic University of America.

If M is the trace in the face ABC of the internal axis of the trihedron $D-ABC$ (see Court's *Modern Pure Solid Geometry*, p. 37, Art. 113, Macmillan, 1935), we have

$$\text{(1)} \qquad \text{vol. } MDBC : \text{area } DBC = \text{vol. } MDAB : \text{area } DAB$$
$$\text{vol. } MDCA : \text{area } DCA$$ and

$$\text{(2)} \qquad \text{vol. } DMBC : \text{area } MBC = \text{vol. } DMAB : \text{area } MAB = \text{vol. } DMCA : \text{area } MCA.$$

Dividing (1) by (2) we obtain the required proportion, hence the trace M satisfies the required condition.

Analogous solutions by *L. M. Kelly* and *P. D. Thomas*.

EDITORIAL NOTE. It can be shown that the foot of the internal axis of the tetrahedron $D-ABC$ is the only point inside the triangle ABC, which satisfies the required condition.

Indeed, let M be a point inside the triangle ABC satisfying the required condition and let L be the trace of CM on AB. We have:

$$AL : BL = \text{area } ALC : \text{area } BLC = ALM : BLM$$
$$= (ALC - ALM) : (BLC - BLM) = AMC : BMC = DAC : DBC.$$

Hence the plane $DCML$ bisects the dihedral angle DC of the trihedron $D-ABC$, according to Gergonne's theorem (*ibid.*, p. 71).

Similarly the point M lies in the bisecting planes of the dihedral angles DA and DB. Hence the proposition.

If negative areas are to be considered, the traces in ABC of the external axes of $D-ABC$ furnish three other solutions of the problem.—N. A. C.

No. 553. Proposed by *Julius S. Miller*, Dillard University, New Orleans.

A flexible endless chain of mass M pounds rests on the surface of a smooth sphere of radius R feet, the chain forming a ring of r feet. Find the tension in the chain.

Solution by *James N. Snyder*, Allegheny College, Meadville, Pennsylvania.

Let T be the tension in the chain, N the normal reaction of the sphere per unit length of chain, $M/2\pi r$ the linear density of the chain, and ϕ the angle which the normal to the sphere at a point of contact makes with the plane of the chain. Consider a length ds of the chain which subtends an angle $d\Theta = ds/r$ at the center of the horizontal circle. Then vertically:

$$Mgds/2\pi r = N\cdot ds\cdot \sin\phi, \qquad (\sin\phi = \sqrt{R^2-r^2}/R). \qquad \text{So}$$

$$N = MgR/2\pi r\sqrt{R^2-r^2}. \tag{1}$$

Resolving forces along the normal of the horizontal circle at the midpoint of ds:

$$2T\sin(d\Theta/2) = N\cdot ds\cdot \cos\phi. \tag{2}$$ But

$$\cos\phi = r/R, \tag{3}$$

$$\sin(d\Theta/2) = d\Theta/2 = ds/2r, \tag{4}$$

to infinitesimals of higher order. Putting (3), (4), and (1) in (2) we get:

$$T = Mgr/2\pi\sqrt{R^2-r^2}.$$

Also solved by *Howard Eves*, *A. R. Thomas* and the *Proposer*. Eves and the Proposer call attention to the fact that as r approaches R the tension increases without limit.

F. G. Fender sends the following comment. In connection with the singularity in the solution when $r=R$ the question arises: Can a flexible loop be broken just by resting it on a sphere of approriate

radius? The above solution implies an affirmative answer, but two facts have been neglected which are never negligible in the case of large forces: (1) Deformation of the object (here, stretching of the loop and contraction of the sphere); (2) Friction. In the present case we may consider deformation as adding to the friction, and take μ as the coefficient of friction. We have then (in the notation of the above solution), w being the weight of the loop,

$$dw = (Mg/2\pi r)r\Theta d = (\sin\phi + \cos\phi)Nds,$$

$$Td\Theta = (\cos\phi - \sin\phi)Nds.$$

Thus $$\frac{2\pi T}{Mg} = \frac{\cot\phi - \mu}{1+\mu\cot\phi}, \qquad T = \frac{Mg}{2\pi}\cot(\phi+\psi),$$

where $\mu = \tan\psi$. The values of T now have the upper bound

$$Mg\cot\psi/2\pi = Mg/2\pi\mu.$$

When $\phi \geqq \pi/2 - \psi$, the tension in the chain is zero.

No. 557. Proposed by *E. P. Starke*, Rutgers University.

A boy makes two snowballs, one having twice the diameter of the other. He brings them into a warm room and lets them melt. When the larger is half melted, how much is left of the smaller.?

Solution by *Frank Hawthorne*, New Rochelle, N. Y.

Since only the surface of the snowball is exposed to the warm air, let us assume that the rate of melting is proportional to the surface area, which is to say that the rate of decrease of the radius is constant.

When the larger snowball has been reduced to half its volume, the ratio of its radius to its original radius is $\sqrt[3]{4}/2$. In the same time the smaller snowball will have had its radius reduced an equal amount. Hence the ratio of its present radius to its original radius is $\sqrt[3]{4}-1$. The part of the smaller which is left is thus $(\sqrt[3]{4}-1)^3 = .2027$ approximately.

Also solved by *H . E. Bowie* and *Howard Eves*.

No. 561. Proposed by *Julius S. Miller*, Dillard University, New Orleans.

A smooth homogeneous sphere rolls, without slipping, on a plane surface. Find the relationship between the kinetic energy of the upper half and that of the lower half.

Solution by *W. Irwin Thompson*, Los Angeles City College.

If the origin is taken at the point of contact of the sphere and the plane, the equation of the sphere is $r^2+(z-R)^2=R^2$ or

$$r^2=2Rz-z^2,$$

where z is the vertical distance, r is the radius of a horizontal section, and R is the radius of the sphere. A horizontal disk of area πr^2 and mass $M=\pi\rho r^2dz$ has a moment of inertia about an axis through the origin and in the given plane surface equal to $\frac{1}{4}Mr^2+Mz^2$ or

$$dI_x=\tfrac{1}{4}\pi\rho r^4dz+\pi\rho r^2z^2dz.$$

Since the kinetic energy is equal to

$$dE=\tfrac{1}{2}dI_x\omega_x^2,$$

the energy of the upper hemisphere is

$$E_1=\frac{\pi\rho\omega_x^2}{8}\int_R^{2R}(R^2-z^2)^2dz+\frac{\pi\rho\omega_x^2}{2}\int_R^{2R}(R^2z^2-z^4)dz$$

or $E_1=43\pi\rho R^5\omega_x^2/60$ units. The energy of the lower hemisphere is

$$E_2=\frac{\pi\rho\omega_x^2}{8}\int_0^{R}(R^2-z^2)^2dz+\frac{\pi\rho\omega_x^2}{2}\int_0^{R}(R^2z^2-z^4)dz$$

$$=13\pi\rho R^5\omega_x^2/60 \text{ units.}$$

Thus $$E_1/E_2=43/13.$$

Also solved, using the parallel-axis transfer theorem (without integration) to obtain the same result, by *Howard Eves, Frank Hawthorne*, and the *Proposer*.

No. 562. Proposed by *Frank C. Gentry*, University of New Mexico.

A variable triangle has a fixed base, and the difference of the base angles is constant. Find the locus of the nine-point center.

I. Solution by *P. D. Thomas*, U. S. Navy.

Let the given triangle have vertices $A(u,v)$, $B(a,O)$, $C(O,O)$. Then the center of the nine point circle is given by the intersection of the lines.

$$x=(2u+a)/4,$$

(1) $$y=v/r+u(a-u)/4v.$$

The condition that the difference of the base angles be constant is

(2) $$v(2u-a)/[u(a-u)+v^2]=k \quad \text{(constant.}$$

The elimination of the parameters u,v between (1) and (2) leads to the equation

$$y=x/k-a/2k,$$

which represents a straight line passing through the midpoint of the base.

II. Solution by *Howard Eves*, Syracuse University.

Let ABC be the variable triangle, BC the fixed base, angle $B\geq$ angle C. Then the tangent to the nine-point circle at the midpoint, M, of BC makes the angle $B-C$ with the base BC. (See lemma 1, Art. 322, Johnson's *Modern Geometry.*) Since, by hypothesis $B-C$ is constant, the required locus is the straight line through M making an angle of $90°+(B-C)$ with the base BC.

Also solved by *Henry E. Fettis*, and *J. S. Guérin*.

No. 564. Proposed by *Henry E. Fettis*, Dayton, Ohio.

Find the locus of the intersections of the tangents to a conic which make a fixed angle with each other.

Solution by *Howard Eves*, Syracuse University.

The locus of the intersection of two tangents to a curve which are inclined at a fixed angle α with each other is called an *isoptic locus* for the given curve, (the *orthoptic locus* when $\alpha=90°$). In finding the isoptic locus corresponding to α for a conic we consider the three cases according as the conic is an ellipse, hyperbola, or parabola.

Case 1: the ellipse $x^2/a^2+y^2/b^2=1$.

The tangential equation of the ellipse is found to be

(1) $$a^2u^2+b^2v^2=1.$$ Let

(2) $$ux+vy+1=0$$

be the tangential equation of a point P on the isoptic locus. Taking (1) and (2) simultaneously so as to render an equation homogeneous in u and v we find

$$a^2u^2+b^2v^2=u^2x^2+2uvxy+v^2y^2,$$ or

(3) $$m^2(a^2-x^2)+m(2xy)+(b^2-y^2)=0,$$

where $m=-u/v$. The roots m_1 and m_2 of (3) are the slopes of the two tangents through P to the given ellipse. Hence we have

$$\text{(F)} \qquad \tan \alpha = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{2\sqrt{a^2y^2 + b^2x^2 - a^2b^2}}{x^2 + y^2 - a^2 - b^2}, \qquad \text{or}$$

$$\text{(4)} \qquad \tan^2\alpha(x^2 + y^2 - a^2 - b^2)^2 = 4(a^2y^2 + b^2x^2 - a^2b^2),$$

which is the cartesian equation of the required isoptic locus. (This rational form of the equation also includes the locus of the intersection of two tangents which are inclined at angle $180° - \alpha$.)

Case 2: the hyperbola $x^2/a^2 - y^2/b^2 = 1$.

Changing $+b^2$ to $-b^2$ in the above we find

$$\text{(5)} \qquad \tan^2\alpha(x^2 + y^2 - a^2 + b^2)^2 = 4(a^2y^2 - b^2x^2 + a^2b^2).$$

Case 3: the parabola $y^2 = 4ax$.

Here the tangential equation is

$$av^2 - u = 0.$$

Taking this simultaneously with $ux + vy + 1 = 0$ we find

$$av^2 + u(ux + vy) = 0,$$

or

$$m^2x - my + a = 0,$$

where $m = -u/v$. Hence

$$\tan \alpha = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\sqrt{y^2 - 4ax}}{a + x}, \qquad \text{or}$$

$$\text{(6)} \qquad \tan^2\alpha(a + x)^2 = y^2 - 4ax.$$

This last locus is a hyperbola. (It can be shown that the two branches of the hyperbola correspond to the isoptic loci for α and $180° - \alpha$ respectively.)

Note: See Hilton's *Plane Algebraic Curves* (1920), ex. 4 page 175. Observe, in particular, that the orthoptic locus of a central conic is a circle and that of a parabola is a straight line.

Paul D. Thomas in his solution makes use of a formula equivalent to the formula (F) above, as given by C. Smith, *Conic Sections*, p. 182, ex. 54.

Also solved by the *Proposer*.

It may be of interest to observe that the left hand side of (4) or (5) equated to zero represents the orthoptic circle of the respective conic. (John Casey, *Treatise on Analytic Geometry of the Point, Line, Circle, and Conic Sections*, p. 520, ex. 9. Dublin, 1893).

EDITORIAL NOTE. In the case of a central conic (C), let u, v be two diameters of (C) making an angle α, and let u', v' be the diameters conjugate respectively to u, v. The tangents p, p' to (C) at the extremities P, P' of u' are parallel to u, and the tangents q, q' at the extremities Q, Q' of v' are parallel to v, hence the vertices of the parallelogram $pqp'q'$ circumscribed about (C) belong to the required locus.

As the diameters u, v revolve about the center O of (C), they describe two superposed projective pencils, the conjugate diameters u', v' describe pencils involutory to those of u, v, and therefore projective to each other, hence the pairs of points P, P' and Q, Q' describe on (C) two involutions projective to each other, and the same is therefore true of the two pairs of tangents p, p' and q, q'. Thus

$$(pp', \cdots)\overline{\wedge}(u', \cdots)\overline{\wedge}(u \cdots)\overline{\wedge}(v, \cdots)\overline{\wedge}(v', \cdots)\overline{\wedge}(qq', \cdots)$$

i. e., we have two pencils of rays of the second order (pp') and (qq') projective to each other, hence the points of intersection of the lines p, p' with the lines q, q' lie on a quartic curve.—N. A. C.

No. 565. Proposed by *Fred Fender*, Rutgers University.

The equivalent electrical resistance s of two resistance x and y connected in parallel is given by the relation

$$\frac{1}{s}=\frac{1}{x}+\frac{1}{y}, \qquad s,\ x,\ y>0.$$

Determine the solution in positive integers.

I. Solution by *Marion L. Gaines*, University of North Carolina.

Since s, x, y are positive, we have $x>s$ and $y>s$. Let $x=s+u$, $y=s+v$. u and v are positive integers and it follows that

$$s^2=uv.$$

Thus for every s, we have merely to decompose s^2 in every way as a product of two integers u and v. The solutions are then given by s, $x=s+u$, $y=s+v$.

II. Solution by *J. S. Guérin*, Catholic University, Washington, D. C.

Let d be the greatest common divisor of x and y, so that $x=da$, $y=db$, with a and b relatively prime integers. The given relation may now be put in the form

$$s=dab/(a+b).$$

Since $a+b$ is relatively prime to a and also to b, one must have

$d=k(a+b)$ in order that s be integral. Then the required solutions are easily determined as

$$s=kab, \quad x=ka(a+b), \quad y=kb(a+b),$$

where k, a, b are arbitrary except that a and b are relatively prime.

From the above discussion the following principle appears: In order that the product of two numbers be divisible by their sum, it is necessary that the two numbers have a common factor greater than unity.

Also solved by *H. S. Grant*, *Frank Hawthorne*, *A. W. Randall*, *A. R. Thomas*, *P. D. Thomas*, *H. J. Zimmerberg*, and the *Proposer*.

PROPOSALS

No. 579. Proposed by *Howard Eves*, Syracuse University.

Find an expression for the length of an altitude of a tetrahedron in terms of the three edges and the three angles at the vertex through which the altitude is taken.

No. 580. Proposed by *Howard Grossman*, New York City.

Find $\lim_{n\to\infty} \cos\frac{\pi}{3}\cos\frac{\pi}{4}\cos\frac{\pi}{5}\cdots\cos\frac{\pi}{n}$.

(See Kasner's *Mathematics and the Imagination*, p. 311, for an interesting geometric interpretation.)

No. 581. Proposed by *N. A. Court*, University of Oklahoma.

On the line LA' joining the given point L to the centroid A' of the face BCD of a tetrahedron $ABCD$, the point P is taken such that $LP : LA$ is equal to a given constant, positive or negative. Prove that the line AP and its three analogous lines BQ, CR, DS have a point, say U, in common.

Determine the locus of U when L describes a fixed plane.

No. 582. Proposed by *E. P. Starke*, Rutgers University.

Find the smallest three-digit number N such that the three numbers obtained from N by cyclic permutations of its digits are in arithmetic progression.

No. 583. Proposed by *Paul D. Thomas*, U. S. Navy.

Construct a triangle given the lengths of the median, the internal

bisector, and the ex-symmedian isssued from the same vertex. (Note: an ex-symmedian is the segment of a tangent to the circumcircle of a triangle at a vertex between that vertex and the respectively opposite side.)

No. 584. Proposed by *Howard Eves*, Syracuse University.

Show that $\sum_0^n \binom{n}{j}(-1)^j/(j+1) = 1/(n+1)$.

Bibliography and Reviews

Edited by
H. A. Simmons and P. K. Smith

Elementary Statistical Methods. By Helen M. Walker. Henry Holt and Company, New York, 1943. xxv+368 pages.

The author of this work is known to all students of statistics for her very useful *Studies in the History of Statistical Method* published in 1929. It is therefore a pleasure to examine her latest work written from the point of view of one who has examined the subject from its historical origins. As one might expect, many of the chapters begin with one or more pertinent quotations from classical authors, and a number of historical references illuminate the text. As one opens the book, his eye falls first upon the following celebrated quotation from George Sarton, historian of science:

> "I like to think of the constant presence in any sound Republic of two guardian angels; The Statisitician and the Historian of Science. The Statistician keeps his finger on the pulse of Humanity, and gives the necessary warning when things are not as they should be. The Historian ... will not allow Humanity to forget its noblest traditions or to be ungrateful to its greatest benefactors. If the Statistician is like a physician, the Historian is like a priest—the guardian of man's most precious heritage .... Humanity must be protected by the watchful Statistician, and it must be sustained in its newer and bolder efforts by the consciousness of every antecedent effort, to which it owes its culture, its dignity, and its excellence."

The book itself was written for students with a minimum of mathematical training. In fifteen chapters, with very little symbolism, the author carries the reader through the graphical representation of frequency distributions, averages, measures of variability, skewness and kurtosis, the normal distribution, the elementary theory of regression and correlation, and the theory of samples and statistical inference. No formulas are found in the first 75 pages of the text, but in Chapter 6 symbolism in statistics is treated and the student is there introduced to symbols for summation. The author, however, is keenly aware of her difficulties for she says in the Preface: "Experience convinces me that the difficulties of symbolism for the beginner are like the thistle—painful when touched gingerly and harmless when grasped boldly."

The book concludes with five appendices, which make some amends for the scanty mathematical discussions in the text itself. Appendix B, 14 pages in length, is devoted to mathematical notes and proofs. Beginning with a mathematical treatment of variance, this appendix discusses moments, regression coefficients, the binomial expansion, and the properties of the normal curve. In the development of the last topic the methods of calculus are employed. Appendix D is devoted to a few four-place tables, and Appendix E lists 100 statistical formulas. Because of its comprehensive nature, this is one of the most useful sections of the book.

The book contains many examples taken mainly from the field of education, and numerous exercises are provided for the student. The text is principally adapted for those who wish to apply statistics to sampling problems. Such topics as index numbers, time series, curve-fitting, etc. are omitted. The author also does not venture into

the more advanced theories of correlation, since this text is essentially an introduction to statistical methods.

Northwestern University. H. T. DAVIS.

Mathematics Dictionary. Revised edition. By Glenn James and Robert C. James. The Digest Press, Van Nuys, California, 1943. viii+273+46 pages.

It seems well to call the attention of mathematics teachers and students once more to this useful work. The authors have attempted to give not only the correct definitions of the various technical terms connected with elementary mathematics, but by means of condensed explanations and cross references to furnish the student with a comprehensive review of the material pertinent to any particular topic. A brief perusal of the book will convince the reader that they have accomplished this purpose most admirably. Many examples and figures are used to illustrate the theory.

The subject matter covered includes arithmetic, algebra, trigonometry, geometry, the Calculus, navigation, physics, astronomy, mathematics of finance and statistics. There are also included many terms which pertain to courses which usually follow the Calculus although the student of such advanced courses may find many omissions. The authors mention in their preface that they plan the publication of a sequel dealing with terms of higher mathematics. This would be a most welcome supplement to the present volume.

The appendix contains eight numerical tables, a table of denominate numbers, differentiation and integral formulas and a list of mathematical symbols. Although there are a few mistakes and several misprints, these do not prevent the book from being a most valuable reference.

The University of New Mexico. F. C. GENTRY.

Differential and Integral Calculus. By Clyde E. Love. The Macmillan Company, New York, 1943. xv+459 pages+18 pages of tables.

A revision of this text is welcomed by college teachers. Comparisons will be made between this 1943 edition and the latest previous edition of 1934.

In this comparison it is first noted that statements have been made more critically concerning range of variable connected with integrals. The author amplifies his previous treatment devoted to the part the principle square root plays in the integral and the determinantion of the range of the variable. This additional attention is evidenced in Sections 6 and 137. In Section 137 the author closes by the statement: "In an integration involving a square root or other many-valued function, particularly when some of the quantities are negative, watch every detail closely to make sure that in each transformation the right branch is taken." (The author should have said "... any other many-valued function, ... ".)

The basic introductory concepts on functions, limits, and continuity are treated in Chapter I, but the author adopted a procedure pedogogically sound in postponing the definitions and graphs on finite and infinite discontinuities to Chapter VII. The average student is bewildered by the lengthy usual first chapter in calculus texts on functions, limits, and continuity.

The author failed to change his definition of the tangent to a curve at a point. The objectionable feature in this definition lies in the statement "at the instant P' coincides with P," found in Section 16 of the two latest editions. This statement is not pedagogically desirable in that the beginner may be led to think that in the deri-

vative the increment of the independent variable should be placed equal to zero in finding the limit of

$$\frac{\Delta y}{\Delta x}.$$

I should like to call attention to the definition of the tangent to a curve found in recent splendid texts (for high grade students) in the calculus by Dresden, Ettlinger, and Sherwood and Taylor.

In evaluating a calculus text its approach to and space given to the fundamental theorem of the integral calculus must be viewed. In this text the fundamental theorem is stated without proof. In the latest edition the author has rearranged his approach to the fundamental theorem slightly. The development of the formula for the area under a curve is given along with the approach to the fundamental theorem. The area between the curve $y=f(x)$, the x-axis, and the lines $x=a$ and $x=b$ is defined to be

$$\lim_{\Delta x\to 0} \sum_{i=1}^{n} f(x_i)\, x,$$

for which the interval $a \overset{<}{=} x \overset{<}{=} b$ is divided into n subintervals each of width Δx and x_i is known to lie in the ith subinterval, provided the limit exists. Then the fundamental theorem is merely stated imposing the sufficient conditions for the existence of the limit indicated above. For the first course in the calculus for the average student the approach to the summation process in this text is sound. It is largely intuitive, but for the average student the more critical approach to the summation process by a proof of the fundamental theorem based on the real number theory is decidedly bewildering and confusing except for the best students.

The text amplifies the tre tment of centroids and moments of inertia. The physical approach to these topics is excellent.

In many colleges there is not room in the curricula for a course in solid analytic geometry. In the approach to the usual chapters on partial derivatives and multiple integrals the solid analytic geometry is a requisite. It would be especially convenient if a chapter on solid analytic geometry were included in the text.

This text has a pleasing format and is larger than the older edition. The figures are very good. The text is recommended to any instructor or department desiring a change in texts.

Louisiana Polytechnic Institute. P. K. SMITH.

Webster's Biographical Dictionary. G. and C. Merriam Co., Springfield, Mass., 1943. xxxvi + 1696 pages. (With Thumb-Notched Index). $6.50.

This book is said to contain 40,000 biographies of noted men and women of all countries, of all ages, including contemporaries. It emphasizes name pronunciation and is said to be the result of years of preparation by the Editorial Staff of *Webster's New International Dictionary* and to have been planned to provide the greatest amount of information of value to the greatest number of users. While the latest edition of *Who's Who in America* lists 33,893 biographies, the dictionary under review lists only about 40,000 of all ages and of all countries. As an instance of its conciseness in regard to mathematicians, we may note that under the name of E. H. Moore, we find only: "1862-1932. American Mathematician, professor, U. of Chicago (1892-1931)". Some of the biographies are, however, longer.

The selection of the names included in this dictionary was obviously very difficult and constitutes one of its most useful features. For instance, all the members of the *National Academy of Sciences* are included in *Who's Who in America*, while only about one-fourth of these members are found in the book under review. It should not be

assumed that these selections will meet with universal approval, but they direct attention to points of excellence which have frequently been overlooked. The high standards of Webster's dictionaries will doubtless inspire confidence in the serious effort to make the volume under review a useful addition to school libraries. While dragnet works are sometimes useful, wise selections are usually more difficult. There is much that should be forgotten in history, including some textbooks' authors.

University of Illinois. G. A. MILLER.

Mathematical and Physical Principles of Engineering Analysis. By Walter C. Johnson. McGraw-Hill Company, New York, 1944. 360 pages.

The McGraw-Hill Company has done an exceptionally fine job in the publication of this volume. The printing is excellent, the page pleases the eye and the book is easy to handle.

This book, on the whole, seems to be very well written. The problems are abundant and sufficiently well stated. Though there are numerous problems in mechanics, emphasis seems to be placed on electrical engineering problems,—in particular, on the electric current circuits. However, close analogies are drawn between mechanics and electric current which should be helpful to the non-electrical engineer.

The book is divided into eleven chapters The content of some of them follows: Chapter V treats of graphical and other approximate methods of integration and graphical and numerical solutions of differential equations. This should be of much use to the engineer. Chapter VI is a treament of the analytical solution of ordinary differential equations. This elementary work includes the usual exercises for the student, while the solution of Bessel's equation is given. This, of course, introduces power series solutions but with no further formal treatment. Bessel functions are also studied here. Methods of checking equations and results are considered in Chapter VIII. This is supplemented in Chapter IX by the dimensional check of equations. Chapter X concerns itself with Fourier series and their application.

The calculus and college physics are suggested as a prerequisite and possibly some elementary differential equations. However if ordinary differential equations are to be given at all they should have occupied an earlier chapter. There is, of course, no reason why Chapter V or portions of it could not be used first if necessary. An early problem in the text, 2 p. 34, is that of a circular cylinder rolling down an incline plane without slipping. The vertical height is h; find the velocity at the bottom of the incline. Enough preceding theory is given to enable one to set up the equations of motion, a set of simultaneous equations of the second order readily reducible to first order equations. After the elimination of the friction force, one arrives at the equation of kinetic energy and work. This is a simple problem for the student with the proper background, but it will be tough for one without differential equations and some applications to mechanics. Even then the student's initiative could be fostered by more illustrative examples worked in the text.

D'Alembert's principle as an equilibrium principle is no novelty, since it is a mere transposition of terms to one side of the equations of motion. As formulated by European writers or as expressed by Silberstein in his vector mechanics, D'Alembert's principle is the principle of virtual work. This leads directly to Lagrange's and Hamilton's equations of motion of a mechanical system or a rigid body with a fixed point. In generalized coordinates of a mechanical system, the constraints or "lost forces" disappear from the equations of motion. This important aspect of mechanics should, in my opinion, have place in a book of this kind. Hamilton's equations are basic in statistical mechanics which has thermodynamics as its principal result, while the *Ham-*

iltonian in nucleur physics leads to the Schrœdinger equation. In any case, as a minimum requirement, the kinetic potential and its use should be cultivated.

In studying boundary value problems it seems preferable, to the writer, to introduce partial differential equations, Fourier series, and Bessel functions through the solution of special problems and then to follow this with the more abstract theory if desired. The development of the equation of motion of the vibrating string with its Fourier series solution as given in the text might preferably precede the study of Fourier series as such. [In passing, it should be pointed out that, on page 250, to the conditions for the term by term differentiation of a Fourier series should be added the conditions that the function should have the same value at each end of the interval and that its derivative should exist.] The equation of motion of the vibrating circular membrane, with or without radial symmetry, could then be developed. A reduction of the partial differential equation leads to the Bessel equation. Following the solution, a detailed study of Bessel's functions and partial differential equations would then be the order. Such mechanical problems, to the writer's notion, are more understandable and should take precedence over the linear flow of heat problems that are presented in the text. These criticisms or suggestions may, of course, be just a matter of taste.

Among the minor criticisms is the following one for page 17. There we find this: "If the mass of the body were all concentrated at the radius of gyration, ..." This could have no meaning to the uninitiated. However, if an engineer has read the book and worked the problems, I think he should have a very well rounded basis for a certain type of engineering practice. Under a competent teacher, with laboratory or supervisory methods, this could well be accomplished.

University of Michigan. VINCENT C. POOR.